Disproof of Riemann's Functional Equation Using the Poisson Summation Formula

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ABSTRACT

Riemann and others have used the Poisson summation formula to prove Riemann's functional equation. The opposite is actually true, the Poisson summation formula provides a refutation of Riemann's functional equation. It is the purpose of this paper to disprove Riemann's functional equation using the Poisson summation formula.

INTRODUCTION

In the course of studying the Riemann Hypothesis, I came across the Poisson summation formula since it has been used to prove Riemann's functional equation given below

(1) $$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

or simply

$$\Phi(s) = \Phi(1-s).$$

That is, the complex function Φ(s) is equal to its reflection Φ(1-s). Where $s = \sigma + \omega i$ is the complex variable with real and imaginary parts $\Re(s)=\sigma$ and $\Im(s)=\omega$, respectively. If true then at s = ½ + ωi, the function Φ(s) is real due to the Reflection Principle [1].

But due to the zeta function $\zeta(s)$, Φ(s) will only converge absolutely if $\sigma > 1$, and converges conditionally if ½ < σ ≤ 1 and $\omega \neq 0$. In other words, Φ(s) needs σ > ½ in order for it to converge [2]. On the other hand, Φ(1-s) needs σ < ½ in order for it to converge due to $\zeta(1-s)$. Thus Φ(s) and Φ(1-s) have no common points, $D_1 \cap D_2 = \emptyset$, where D_1 is the domain of Φ(s) and D_2 is the domain of Φ(1-s). Furtermore, Φ(s) can't be real at s = ½ + ωi since it is undefined on that region. Hence the two functions are *not* equal $\Phi(s) \neq \Phi(1-s)$, because there are no values of s in which (1) holds.

THE POISSON SUMMATION FORMULA

In studying the Riemann Hypothesis, I've made a mistake of questioning the validity of the Poisson summation formula since I've assumed that (1) is true and that the Poisson summation formula is making it untrue. But after carefullyly examining the Poisson summation formula (I was actually trying to disprove it), I came at the opposite conclusion: the Poisson summation formula is true while Riemann's functional equation is false.

For any appropriate non-periodic function f and its Fouirier transform $\hat{f}$, Poisson summation formula states that

(2) $$\sum_{n=-\infty}^{\infty} f(t+nP) = \sum_{k=-\infty}^{\infty} \frac{1}{P}\hat{f}\left(\frac{k}{p}\right)e^{\frac{2\pi i k t}{P}}.$$

The left-side of (2) is a periodic summation resulting in a periodic function $f_P(t)$ with period P, that is

$$\sum_{n=-\infty}^{\infty} f(t+nP) = f_P(t) = f_P(t+nP).$$

While the right-side of (2) is its Fourier series representation, that is

$$\sum_{k=-\infty}^{\infty} \frac{1}{P}\hat{f}\left(\frac{k}{p}\right)e^{\frac{2\pi ikt}{P}} = \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi ikt}{P}},$$

where $\{c_k\}$ are the Fouirier coefficients

$$c_k = \frac{1}{P}\int_{-P/2}^{P/2} f_P(t)e^{\frac{-2\pi ikt}{P}}dt = \frac{1}{P}\int_{-P/2}^{P/2}\left\{\sum_{n=-\infty}^{\infty} f(t+nP)\right\}e^{\frac{-2\pi ikt}{P}}dt,$$

let $\tau = t + nP$ so that $d\tau = dt$; since t is from $-P/2$ to $P/2$, τ is now from $(n - 1/2)P$ to $(n+1/2)P$ and interchange the infinite summation and integration

$$c_k = \frac{1}{P}\sum_{n=-\infty}^{\infty}\left\{\int_{(n-1/2)P}^{(n+1/2)P} f(\tau)e^{\frac{-2\pi ik(\tau-nP)}{P}}dt\right\} = \frac{1}{P}\int_{-\infty}^{\infty} f(\tau)e^{\frac{-2\pi ik\tau}{P}}e^{2\pi ikn}d\tau,$$

since $e^{2\pi ikn} = 1$ for all k and n, and $\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\nu t}dt$:

$$c_k = \frac{1}{P}\hat{f}\left(\frac{k}{P}\right).$$

If the non-periodic function is

$$f(t) = e^{-\pi xt^2} \qquad x > 0 \text{ and } -\infty < t < \infty,$$

its Fourier transform is

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\nu t}dt = \int_{-\infty}^{\infty} e^{-\pi xt^2}e^{-2\pi i\nu t}dt$$

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} e^{-\pi x\left(t^2 + \frac{2i\nu t}{x}\right)}dt = \int_{-\infty}^{\infty} e^{-\pi x\left(t^2 + \frac{2i\nu t}{x} + \left(\frac{i\nu}{x}\right)^2 + \frac{\nu^2}{x^2}\right)}dt$$

$$\hat{f}(\nu) = e^{-\frac{\pi\nu^2}{x}}\int_{-\infty}^{\infty} e^{-\pi x\left(t + \frac{i\nu}{x}\right)^2}dt,$$

let $u = t + \frac{i\nu}{x}$, then $du = dt$,

$$\hat{f}(\nu) = e^{-\frac{\pi\nu^2}{x}}\int_{-\infty}^{\infty} e^{-\pi xu^2}du$$

$$\left(\hat{f}(\nu)\right)^2 = \left(e^{-\frac{\pi\nu^2}{x}}\int_{-\infty}^{\infty} e^{-\pi x u^2}du\right)\left(e^{-\frac{\pi\nu^2}{x}}\int_{-\infty}^{\infty} e^{-\pi x y^2}dy\right) = \left(e^{-\frac{2\pi\nu^2}{x}}\int_{0}^{\infty}\int_{0}^{2\pi} e^{-\pi x r^2} r dr d\theta\right)$$

$$\hat{f}(\nu) = \frac{1}{\sqrt{x}}\, e^{-\frac{\pi\nu^2}{x}} \qquad x > 0 \text{ and } -\infty < \nu < \infty.$$

The Poisson summation formula becomes

$$\sum_{n=-\infty}^{\infty} e^{-\pi x(t+nP)^2} = \frac{1}{P}\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{x}}\, e^{-\frac{\pi}{x}(k/P)^2}\, e^{\frac{2\pi i k t}{P}},$$

if $P = 1$ and $t = 0$

(3)
$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{k=-\infty}^{\infty} e^{-\pi\frac{k^2}{x}} \qquad x > 0.$$

Letting

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \qquad x > 0,$$

the equality in (3) becomes

(4)
$$1 + 2\,\psi(x) = \frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right) \qquad x > 0,$$

The integrals on both sides of (4) from 0 to ∞ are infinite, that is

$$\int_0^{\infty}\left[1 + 2\,\psi(x)\right]dx = \int_0^{\infty}\left\{\frac{1}{\sqrt{x}} + 2\frac{1}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right)\right\}dx = \infty,$$

because

$$\int_0^{\infty} 1\,dx = \infty, \qquad \int_0^{\infty}\frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_0^{\infty} = \infty,$$

$$\int_0^{\infty}\frac{2}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right)dx = \int_0^{\infty}\frac{2}{\sqrt{x}}\sum_{k=1}^{\infty} e^{-\pi\frac{k^2}{x}}dx,$$

let $y = \dfrac{\pi k^2}{x}$, $dy = -\pi k^2 x^{-2}dx$, such that $x = \dfrac{\pi k^2}{y}$, and $dx = -\dfrac{x^2}{\pi k^2}dy$. If x is from 0 to ∞, then, y is from ∞ to 0:

$$= 2\sum_{k=1}^{\infty}\left\{\int_{\infty}^{0}\sqrt{\frac{y}{\pi k^2}}\ e^{-y}\ \left\{-\left(\frac{\pi k^2}{y}\right)^2\frac{dy}{\pi k^2}\right\}\right\}$$

$$= 2\pi^{1/2}\left(\sum_{k=1}^{\infty}k\right)\left(\int_{0}^{\infty}y^{-3/2}e^{-y}dy\right) = \infty,$$

because $\sum_{k=1}^{\infty}k = \infty$ and $\int_{0}^{\infty}y^{-3/2}e^{-y}dy = \infty$. The only finite integral is

$$2\int_{0}^{\infty}\psi(x)dx = 2\int_{0}^{\infty}\left(\sum_{n=1}^{\infty}e^{-\pi n^2 x}\right)dx = 2\sum_{n=1}^{\infty}\left(\frac{e^{-\pi n^2 x}}{(-\pi n^2)}\right)\Bigg|_{0}^{\infty} = 2\sum_{n=1}^{\infty}\frac{1}{\pi n^2} = \frac{\pi}{3}.$$

RIEMANN'S FUNCTIONAL EQUATION IS NOT VALID

Riemann's functional equation can also be expressed as

(5) $$\int_{0}^{\infty}x^{\frac{s}{2}-1}\psi(x)dx = \int_{0}^{\infty}x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx,$$

and from (4) one obtains

(6) $$\psi(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right).$$

Substitute (6) into $\Phi(s)$

$$\Phi(s) = \int_{0}^{\infty}x^{\frac{s}{2}-1}\psi(x)dx = \int_{0}^{\infty}x^{\frac{s}{2}-1}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)\right)dx$$

$$\Phi(s) = \frac{1}{2}\int_{0}^{\infty}x^{\frac{s-3}{2}}dx - \frac{1}{2}\int_{0}^{\infty}x^{\frac{s}{2}-1}dx + \int_{0}^{\infty}x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx.$$

Let $x = x^{-1}$ so that $dx = x^{-2}dx$ for the last integral above

$$\Phi(s) = \frac{x^{\frac{s-1}{2}}}{s-1}\Bigg|_{0}^{\infty} - \frac{x^{\frac{s}{2}}}{s}\Bigg|_{0}^{\infty} + \left\{\int_{\infty}^{0}x^{\frac{-s+3}{2}}\psi(x)(-x^{-2}dx) = \int_{0}^{\infty}x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx\right\}$$

$$\Phi(s) = \left.\frac{x^{\frac{s-1}{2}}}{s-1}\right|_0^\infty - \left.\frac{x^{\frac{s}{2}}}{s}\right|_0^\infty + \Phi(1-s) = \infty - \infty + \Phi(1-s) = \text{undefined}.$$

$\Phi(s)$ becomes undefined if one applies the Poisson summation formula on it. The reason for this is that one should not use (6) to obtain the integral if the limits are from 0 to ∞ and one should remove the terms that is making the integral undefined,

$$\int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx = \int_0^\infty x^{\frac{s}{2}-1}\frac{1}{2}\left\{\frac{1}{\sqrt{x}} - 1 + \frac{2}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right)\right\}dx$$

$$\int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx = \int_0^\infty x^{\frac{s}{2}-1}\frac{1}{2}\left\{\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right) - 1\right\}dx = \int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx$$

$$\Phi(s) = \Phi(s).$$

DISPROOF FOR RIEMANN'S FUNCTIONAL EQUATION

(7) $$\Phi(s) = \int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx = \int_0^1 x^{\frac{s}{2}-1}\,\psi(x)dx + \int_1^\infty x^{\frac{s}{2}-1}\,\psi(x)dx,$$

Substituting formula (6) on the right-side of (7) for the integral from x = 0 to 1:

(8) $$\int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx = \int_0^1 x^{\frac{s}{2}-1}\left\{\frac{1}{2\sqrt{x}} + \frac{1}{2} + \frac{1}{\sqrt{x}}\,\psi\left(\frac{1}{x}\right)\right\}dx + \int_1^\infty x^{\frac{s}{2}-1}\,\psi(x)dx$$

$$\int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx = \frac{1}{s(s-1)} + \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\,\psi\left(\frac{1}{x}\right)dx + \int_1^\infty x^{\frac{s}{2}-1}\,\psi(x)dx$$

Let $x = x^{-1}$, then $dx = -x^{-2}dx$:

(9) $$\int_0^\infty x^{\frac{s}{2}-1}\,\psi(x)dx = \frac{1}{s(s-1)} + \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\,\psi(x)dx + \int_1^\infty x^{\frac{s}{2}-1}\,\psi(x)dx$$

At this point, Riemann substituted $s = ½ + \omega i$ on (9) without evaluating the other integrals on the right-side obtaining

(10) $$\int_0^\infty x^{-\frac{3}{4}+\frac{\omega i}{2}}\,\psi(x)dx \neq \frac{1}{(-1/4-\omega^2)} + \int_1^\infty x^{-3/4}\cos\left(\frac{\omega}{2}\log x\right)\psi(x)dx.$$

The left-side integral on (10) is a complex quantity while the right-side is real! Thus, the Poisson summation formula disproves that $\Phi(s)$ is real at s = ½ + ωi. This is because Riemann did not remove the terms that is making the integral on the right-side of (8) undefined. The integral

$$\int_0^1 x^{\frac{s}{2}-1}\left\{\frac{1}{2\sqrt{x}}+\frac{1}{2}+\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)\right\}dx,$$

still contains terms that is making the integral on the right-side of (8) undefined and must be remove from it. This leads again to

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_0^\infty x^{\frac{s}{2}-1}\frac{1}{2}\left\{\frac{1}{\sqrt{x}}+\frac{2}{\sqrt{x}}\psi\left(\frac{1}{x}\right)-1\right\}dx=\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$\Phi(s)=\Phi(s).$$

Also, the two integrals on the right-side of (9) are *both* undefined at s = ½ + ωi,

$$\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx \qquad \Re(s)<1/2 \quad \text{and} \quad \int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx \qquad \Re(s)>1/2,$$

since

$$\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx=\Phi(1-s)-\int_0^1 x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx \qquad \Re(s)<1/2, \quad \text{and}$$

$$\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx = \Phi(s)-\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx \qquad \Re(s)>1/2.$$

CONCLUSION

The reason why Riemann's functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \neq \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

is because Riemann obtained it from the equation

(11) $$\zeta(s) \neq 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

The function on the right-side of (11) is *not* a valid function since (11) was obtained from misapplying the Residue theorem [1] on the contour integral

$$\int_C (-z)^{s-1} e^{-z} dz \qquad \Re(z) > 0 \text{ and } \Re(s) > 0.$$

The Poisson summation formula doesn't prove Riemann's functional equation but only shows that $\Phi(s) = \Phi(s)$ and that at $s = ½ + \omega i$ we have an invalid equality which *disproves* $\Phi(½ + \omega i) = \Phi(½ - \omega i)$.

REFERENCES:

[1] Evangelista, Armando M. (2018). *Riemann's Functional Equation is Not Valid and its Implication on the Riemann Hypothesis*. http://vixra.org/abs/1808.0624

[2] Evangelista, Armando M. (2018). *The Domain of the Riemann Zeta Function on the Complex Plane*. http://vixra.org/abs/1808.0684

[3] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity.*

LINKS:

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis
- https://en.wikipedia.org/wiki/Poisson_summation_formula