Distribution of primes applied to sieves and consequence to twin primes conjecture

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Abstract
This paper gives us an application of Eratosthenes sieve to distribution mean distance between primes using first and upper orders of Gauss integral logarithm $Li(x)$.We define function Υ in section 5. Sections $1-4$ give us an introduction to the terminology and a clarification on Υ terms. Section 6 reassumes foregoing explanations and gives us two theorems using first and upper integral logarithm orders.

1. Introduction

A number is a *twin* if it can be found at distance two from the previous or following number of a sequence.
A famous algorithm for making table of primes is the sieve of Eratosthenes: sequentially write down the integers from 2 to a number n that is the last of table; cross out all number greater then 2 which are divisible by 2; the remaining numbers are all twins but only some of them will survive to the next deletions. Then we can find the smallest remaining number greater then 2: it is 3. So we cross out all numbers greater then 3 which are divisible by 3,the remaining ones are twins in the form $6k+1$ and $6k-1$. As previous only some of them will survive to the next deletions. We can go as far as $\lfloor\sqrt{n}\rfloor$ so the numbers remaining are prime.
A famous conjecture affirms that there are an infinite number of twin primes. We can demonstrate this conjecture by the natural distribution of primes using the formula $\frac{n}{\ln n}$, discovered by Gauss and upper orders of $Li(x)$ function (integral logarithm) discovered by Gauss too.

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2. Inaccurate sieves

First we observe that, if a number x is prime, it will suffice to control if it is divisible by integers under $\sqrt{x}$. So, if we want to realize an exact sieve of Eratosthenes to the number x we must cross out multiples of primes numerically inferior to $\sqrt{x}$.
We name *would-be twin primes* those numbers in the form $6k+1$, $6k-1$ included between x and its root. They are

$$\lfloor \frac{2x}{6} \rfloor - \lfloor \frac{2\sqrt{x}}{6} \rfloor \tag{1}$$

using (1) we can miscalculate two numbers in excess at most but this is a negligible error. To find the couples, it will suffice to divide by two.
Primes greater then 3, under $\sqrt{x}$, will be named *eliminators* because they will surely cut out some of the would-be twin primes thanks to the sieve; in fact these numbers, being $6k+1$ or $6k-1$, have a period of 6. This means that they have to "jump" six times to complete the period; so

$$6k+1 \equiv 1 \ mod \ 6$$

$$6k-1 \equiv -1 \ mod \ 6$$

The following figure clarifies the terminology

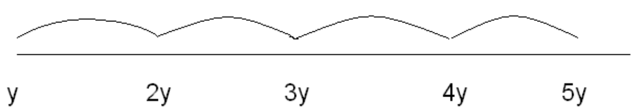

Fig 1 An eliminator y and its "jumps"(or multiples).

In the figure, y is an eliminator and the "jumps" are its multiples. Specifically the eliminator $6k+1$, after 4 jumps, cuts out a $6k-1$ number and, after 6 jumps, it cuts out a $6k+1$ number; $6k-1$ behaves specularly.
The very natural conclusion is that every y eliminator will take x/y jumps: $x/6y$ will delete numbers of its own form and $x/6y$ will cut out numbers of the opposite one. An y eliminator will delete totally:

$$\lfloor \frac{x}{6y} \rfloor + \lfloor \frac{x}{6y} \rfloor \sim \lfloor \frac{x}{3y} \rfloor \tag{2}$$

would-be twin primes. Among these we can reject the jumps within the eliminator's zone, so we consider only the would-be twin primes over $\sqrt{x}$. Finally the formula (2) becomes

$$K = \lfloor \frac{x}{3y} \rfloor - \lfloor \frac{\sqrt{x}}{3y} \rfloor \tag{3}$$

which doesn't consider possible repetitions, i.e. two or more eliminators could delete the same would-be twin prime,but, by this way, we consider two or more times the same deleted number; moreover we conjecture that whenever an eliminator makes a jump it deletes a pair of twins, but actually it could delete a number already deprived of its twin (i.e. it is insignificant in the amount of couples).
The final formula is

$$\lfloor \frac{x}{6} \rfloor - \lfloor \frac{\sqrt{x}}{6} \rfloor - \sum_{y \in \pi(\sqrt{x})} K \tag{4}$$

with K is (3) and $\pi(\sqrt{x})$ are primes under $\sqrt{x}$. Thinking that (4) verifies the conjecture (i.e. it is positive) is an utopia, unless we presume that the distribution of eliminators is near $\sqrt{x}$. We have only to refine the estimate, by removing repetitions and introducing new remarks.

3. The w(n) function

Now we can introduce the $w(n)$ function that represent the number of distinct prime factors of a number n. Hence we can avoid repetitions, in fact all numbers hit by an eliminator are deleted necessarily by another eliminator too (because of factorization); so we can easily halve the number of jumps. But a would-be twin primes could be deleted by more than two numbers, for example if all numbers would be hit by three eliminators we should divide the total amount of jumps by three. All we need now is a function that shows how many different primes factor a number. Now we consider mean increase of $w(n)$ (it is be considered that the growth of $w(n)$ is very irregular). This is the number eliminators jumps should be divided by.

$$L = \frac{1}{\#z} \sum_{z \in [1,n]} w(z) \tag{5}$$

4. Completely deleted twins

Using the prime numbers theorem:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln(x)} = 1$$

and according to Chebyschev boundes

$$0.92...\frac{x}{\ln(x)} \leq \pi(x) \leq 1.105...\frac{x}{\ln(x)}$$

we obtain this assertion: if we have x numbers (considering that $\frac{x}{\ln(x)}$ are primes) we can choose as mean distance among close primes precisely $\ln(x)$. In fact this is a weak form (first order expansion) of integral logarithm $Li(x)$:

$$Li(x) = \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k}$$

Littlewood (1914) demonstrated that $Li(x) - \pi(x)$ change sign infinity times so $Li(x)$ is a good would-be mean, yet to simplify calculus we are going to use $\frac{x}{\ln(x)}$ (first order approximation) reserving upper orders to final theorem. When the distance between two primes, that according the foregoing estimate is

$$\mu - x = \ln(x), \tag{6}$$

increases as far as it becomes bigger then 8, we know that between two primes there is a completely deleted couple of would-be twin primes (remembering that period is 6) i.e. a couple of twin doesn't survive eliminator's jumps. This is a good thing because, thanks to this sacrifice, the other couples of primes will survive proving our claim. As a matter of fact, if a couple is completely erased, this means that we needed to use two jumps to eliminate it. So we take into account the jumps that will hit numbers already deprived of their twin, from the whole number of jumps. These jumps, in fact, are irrelevant to count the would-be couples of twin that will be deleted (indeed, we can consider a couple of twin already destroyed, from the moment it misses one of the two members).
Mathematically we can write

$$T = \sum_{p\in[\sqrt{x},x]} \lfloor \frac{dist(p) - 3}{6} \rfloor \tag{7}$$

where $dist(p)$ gives distance function between p (prime) and the following prime number. Now

$$\tilde{T} = \sum_{p=\sqrt{n}}^{n} (\lfloor \frac{\ln(p) - 3}{6} \rfloor)^{+} \tag{8}$$

is the number of jumps needed to completely erase the twin couples between $\sqrt{n}$ and n. Every x in (6) is, in fact, a prime hence if we develop the formula above we obtain

$$\sum_{p=\sqrt{n}}^{n} \lfloor \frac{\ln(p)}{6} \rfloor - \lfloor \frac{3}{6} (\frac{n}{\ln(n)} - \frac{\sqrt{n}}{\ln \sqrt{n}}) \rfloor \sim \sum_{p=\sqrt{n}}^{n} \frac{ln(p)}{6} - \frac{5.5}{6} (\frac{n}{\ln(n)} - \frac{\sqrt{n}}{\ln \sqrt{n}}) \tag{9}$$

clearly we added here floor functions so we have to take away at least a number of elements equal to decimals you may obtain from formula above. For this considering numerator divided by 6 the following decimals can be obtained: $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$; adding these numbers we obtain 2.5, thus we have to add to the numerator $-2.5 * (\frac{n}{\ln(n)} - \frac{\sqrt{n}}{\ln \sqrt{n}})$: this clarifies 5.5.

5. Alternative approach to use of w(n) function and L term

Considering would-be twin primes between x and its root, we find that these numbers are $\lfloor \frac{2x}{6} \rfloor - \lfloor \frac{2\sqrt{x}}{6} \rfloor$ according to (1). If we cut out from this formula the primes up to $[\sqrt{x}, x]$ we get the primes erased by the sieve without repetitions. Hence we obtain effortlessly the (5) and third member of (4). Therefore

$$\frac{1}{L} \sum_{y \in \pi(\sqrt{x})} K = \lfloor \frac{x}{3} \rfloor - \lfloor \frac{\sqrt{x}}{3} \rfloor - \pi(x) + \pi \sqrt{(x)} \tag{10}$$

without computing $w(n)$ for each would-be prime numbers. Therefore using all these approximations we can write the twin primes computing function :

$$\Upsilon(x) = \lfloor \frac{x}{6} \rfloor - \lfloor \frac{\sqrt{x}}{6} \rfloor - \frac{1}{L} \sum_{y \in \pi(\sqrt{x})} K + T \tag{11}$$

with

$$\frac{1}{L} \sum_{y \in \pi(\sqrt{x})} K = \lfloor \frac{x}{3} \rfloor - \lfloor \frac{\sqrt{x}}{3} \rfloor - \pi(x) + \pi \sqrt{x} \tag{12}$$

$$T = \sum_{p \in [\sqrt{x}, x]} \lfloor \frac{dist(p) - 3}{6} \rfloor \tag{13}$$

It represents the amount of twin primes included between $\sqrt{x}$ and x; if this amount is bigger then zero, our conjecture (affirming that *there are endless twin primes couples*) is proved.

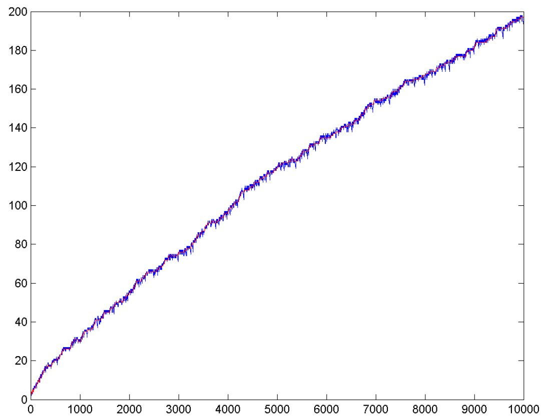

Fig. 2 With:
Red $\rightarrow$ exact twin primes function into $[\sqrt{x}, x]$ and
Blue $\rightarrow$ $\Upsilon(x)$.

Also, using prime number theorem:

$$\pi(x) - \pi\sqrt{x} \sim (\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}). \tag{14}$$

6. Approximation of twin primes'counting function

Hence considering the previous conclusions we reach the following approximation

$$\tilde{\Upsilon}(x) = \lfloor\frac{x}{6}\rfloor - \lfloor\frac{\sqrt{x}}{6}\rfloor - \frac{1}{\tilde{\tilde{L}}}\sum_{\tilde{y}\in\pi(\sqrt{x})}\tilde{K} + \tilde{T} \tag{15}$$

with

$$\frac{1}{\tilde{\tilde{L}}}\sum_{\tilde{y}\in\pi(\sqrt{x})}\tilde{K} = \lfloor\frac{x}{3}\rfloor - \lfloor\frac{\sqrt{x}}{3}\rfloor - (\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) \tag{16}$$

$$\tilde{T} = \frac{(\ln(x) - 5.5)}{6}(\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) \tag{17}$$

Teorema 1 *The function $\tilde{\Upsilon}(x)$, defined by (15)-(17), is bigger then zero for each $x > 141.83$. This function approximates $\Upsilon(x)$, the twin primes counting function, defined by (11)-(13).*

$$\frac{\sqrt{x}-x}{6} + (\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) + \frac{(\ln(x)-5.5)}{6}(\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) \geq 0$$

$$\frac{\sqrt{x}-x}{6} + \frac{\ln(x)}{6}(\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) + \frac{0.5}{6}(\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) \geq 0$$

using logarithms properties

$$\frac{\sqrt{x}-x}{6} + (\frac{x}{6} - \frac{2\sqrt{x}}{6}) + \frac{0.5}{6}(\frac{x}{\ln(x)} - \frac{2\sqrt{x}}{\ln x}) \geq 0$$

$$\frac{0.5}{6}(\frac{x-2\sqrt{x}}{\ln(x)}) - \frac{\sqrt{x}}{6} \geq 0$$

least common multiple

$$\frac{\sqrt{x}(0.5\sqrt{x}-1-\ln(x))}{\ln x} \geq 0$$

So the denominator is bigger then zero for each $x > 1$, and the numerator is a logarithmic equation having as solutions 0.52 and 141.83. This equation is positive for external values of the following range: $[0.52, 141.83]$. Hence $\tilde{\Upsilon}(x)$ is positive for each $x \in [0.52, 1] \cup [141.83, \infty)$ proving our theorem.
$\diamond$.
For each $x < 20$ you may count the twin primes by hand; the theorem assures that our conjecture is proved, because for each new x, the twins will increase more and more, assuming that the function is always positive. This assumption guarantees that twin primes are endless.
The very basic computing hypothesis to demonstrate the theorem is the assumption that the distance among primes is, in mean, bigger or equal then $\ln(x)$: the same statement is valid for the estimate of J function in (27).If we assumed that the distance among primes were $k\ln(x)$ with $0 < k < 1$ from Gauss'approximation our theorem had not been proved. For instance, with $k = 0.7851$ we would have:

$$\frac{\sqrt{x}(\sqrt{x}0.5 - 1 - 0.5702\ln(x) - 0.2149\sqrt{x}\ln(x))}{\ln(x)} \geq 0$$

$$\frac{\sqrt{x}(\sqrt{x}(0.5 - 0.2149\ln(x)) - 1 - 0.5702\ln(x))}{\ln(x)} \geq 0$$

So, positiveness of numerator depends from the following expression

$$\sqrt{x}(0.5 - 0.2149\ln(x)) \tag{18}$$

This quantity is positive at the very beginning, but it will become soon negative because of $\ln(x)$, by this way, the numerator will become negative too, considering that we will find a sum of all negative terms; and this fact will reject our conjecture.
Yet using $k\ln(x)$ with $0 < k < 1$ like mean distance between twin primes we are assuming $\pi(x) \sim \frac{1}{k}\frac{x}{\ln x} \sim Li(x)$ for some upper orders depending on k choice.

Teorema 2 *Using hypothesis from theorem (1) but using like mean distance*

$$\frac{1}{k}\frac{x}{\ln(x)}$$

with $0 < k < 1$ *the approximating function* $\tilde{\Upsilon}(x)$ *is positive for each* $x >$ 141.83

Following the step of the foregoing demonstration we have

$$\frac{\sqrt{x}-x}{6} + \frac{1}{k}(\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) + \frac{1}{k}\frac{(k\ln(x) - 5.5)}{6}(\frac{x}{\ln(x)} - \frac{\sqrt{x}}{\ln\sqrt{x}}) \geq 0$$

from which

$$\frac{1}{k}\frac{0.5}{6}(\frac{x-2\sqrt{x}}{\ln(x)}) - \frac{\sqrt{x}}{6} \geq 0$$

and so

$$\frac{\sqrt{x}(\frac{0.5}{k}\sqrt{x} - \frac{1}{k} - \ln(x))}{\ln x} \geq 0$$

check hypothesis because the numerator is a logarithm function like theorem (1) one. $\diamond$
The conjecture has been demonstrated to all orders of $Li(x)$ function, in fact observing its expansion we are assuming:

$$\frac{1}{k} = \sum_{t=0}^{\infty}\frac{t!}{(\ln x)^t}$$

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