# THE KAKEYA SET CONJECTURE IS TRUE

J.ASPEGREN

Abstract. In this article we will prove the Kakeya set conjecture. In addition we will prove that in the usual approach to the Kakeya maximal function conjecture we can assume that the tube-sets are maximal. Third, we build a direct connection between line incidence theorems and Kakeya type conjectures.

## 1. Introduction

The Kakeya maximal function conjecture and it's variations have gained considerable interest especially after an influential paper by Bourgain (1). For example, it would follow from the conjecture that the Kakeya sets and the Nikodym sets have necessarily full dimensions (10; 11; 7). However, the Nikodym set conjecture is implied by the Kakeya set conjecture (7; 11).The case $n = 2$ was proved by Davies see (4) and the finite field case by Dvir (5). A Kakeya is a set that contains an unit line in every direction. For surveys see (15; 12; 2). Almost all the necessary preliminaries for this paper can be found for example in (7), (10) and in (13). Define the $\delta$ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\omega^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot\omega| \le \frac{1}{2}, |proj_{\omega^\perp}(x-a)| \le \delta\}.$$

Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. Define the Kakeya maximal function $f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a\in\mathbb{R}^n} \frac{1}{T_\omega^\delta(a)} \int_{T_\omega^\delta(a)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension $n$. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f_\delta^*||_p \le C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1}$$

for all $\epsilon > 0$. Remarkably, a bound of the form (1) follows from a bound of the form

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)} \le C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{2}$$

for all $\epsilon > 0$, and for any set of $\delta$-separated of $\delta$ - tubes. See for example (11) or (7). We will prove that we need to consider only the case were the set $\Omega$ is maximal. As usual we define that "$A \lessapprox B$ iff for all $\epsilon > 0$ and for all $\delta > 0$, it holds that $A \le C_\epsilon \delta^{-\epsilon} B$.

1991 *Mathematics Subject Classification.* 42B37,28A75.
*Key words and phrases.* Kakeya conjectures.

We define the Minkowski dimension $Dim_M(K)$ of any bounded set $K$ as follows. Let $\delta > 0$ and let $K_\delta$ be the $\delta-$ neighbourhood of the set $K$, then

$$Dim_M(K) = n - \lim_{\delta \to 0} \frac{\log |K_\delta|}{\log \delta},$$

if the limit exists.
We define the (spherical) Hausdorff content $H^s(K)$ of a subset $K \subset \mathbb{R}^n$ as follows. Let $r > 0$ and let $0 < r_j < r$. Then we define

$$H^s_r(K) = \inf\{\sum_{j=i}^{\infty} r_j^s | K \subset \bigcup_{j=1}^{\infty} B(x_j, \frac{r_j}{2})\},$$

where each $B(x_j, \frac{r_j}{2})$ is a ball with a diameter strictly less than $r$. The $s-$ dimensional Hausdorff content of $K$ is defined as $\lim_{r\to 0} H^s_r(K)$. We define the Hausdorff dimension as

$$Dim_H(K) = \inf\{s \geq 0 | H^s(K) = 0\}.$$

We will prove that

**Theorem 1.** *Every Kakeya set has full dimension.*

## 2. A reduction to the case where the tube-sets are maximal

Let $\Omega'$ be any set of $\delta$ - separated directions. We will prove that

$$|| \sum_{\omega' \in \Omega'} 1_{T'_\omega(a_{\omega'})} ||_{p/(p-1)} \leq || \sum_{\omega \in \Omega} 1_{T_\omega(a_\omega)} ||_{p/(p-1)},$$

where $\Omega$ is maximal. We construct the set $\Omega$ as follows. Let $\Omega'$ be the original direction-set and let $\Omega' \subset \Omega'''$ be maximal. Define

$$\Omega'' := \{\omega'' \in S^{n-1} | \omega'' \in \Omega''' / \{\Omega'\}\}.$$

Moreover, let

$$\Omega := \Omega' \cup \Omega''.$$

Clearly, $\Omega$ is maximal. We choose the tubes corresponding to directions in $\Omega'$ to have origo as their center of masses. Thus, what we do is that we add tubes to the original tube-set so it becomes maximal. Now, we can estimate:

$$|| \sum_{\omega' \in \Omega'} 1_{T'_\omega(a_{\omega'})} ||_{p/(p-1)} \leq || \sum_{\omega' \in \Omega'} 1_{T'_\omega(a_{\omega'})} + \sum_{\omega'' \in \Omega''} 1_{T''_\omega(0)} ||_{p/(p-1)}$$
$$= || \sum_{\omega \in \Omega} 1_{T_\omega(a_\omega)} ||_{p/(p-1)}.$$

Thus, we need only to consider the cases where the tube sets are maximal.

## 3. Previously known results

We will use the following bound for the pairwise intersections of $\delta$ - tubes:

**Lemma 1** (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in (7).

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, dene its $\delta$-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an $\delta-$separated subset of $\Omega$.

**Lemma 2.** *In the notation just defined*

$$1 \leq N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in (7).

## 4. A line intersection theorem and a lemma

We consider a set of lines $L$ where no two lines belong to a common plane containing say, $A$. Because the number of lines is finite we can always find such an point $A$. Moreover we assume that each line intersects only once. We will prove a bound for the number line intersections. The order of intersections is how many lines intersect in a intersection point.

**Theorem 2.** *Let $L$ be a finite set lines, where each line intersects only once. The number of intersections of order $k$ of $n$ lines, $|S_k|$, is less than $\sim n/k$.*

*Proof.* Let $A$ be a point that does not belong to any plane containing two lines from the set $L$. We form a set of distinct planes $S$, where each intersection belongs to $\sim k$ unique planes and no plane contains two or more intersections. This is possible because the number of intersections is finite. We can choose a suitable point $c_i$ from a line intersecting $p_i$ in a way that no other intersection point $p_j$ belongs to the plane defined by points $A$, p$_i$ and $c_i$. In total we have then less than $n$ planes in $S$. So the number of of intersections of order $k$ is less than $\sim n/k$. ☐

We define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{\omega \in \Omega} 1_{T_\omega^\delta}(x) \leq 2^{k+1}\}.$$

Now, the number of the sets $E_{2^k}$ is less than $C_n \log(\delta^{-1})$.

**Lemma 3.** *It holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1} 2^{-k/(n-1)}. \tag{3}$$

*Proof.* Let us suppose that $2^k \sim \delta^{-\beta}, 0 < \beta \leq n-1$, and let‘s suppose that tube $T_{\omega'}$ intersecting $T_\omega \cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between $T_\omega$ and $T_{\omega'}$ is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 1 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \leq |T_\omega \cap T_{\omega'} \cap E_{2^k}| \leq |T_\omega \cap T_{\omega'}| \lesssim \delta^{n-1+\beta/(n-1)} \sim \delta^{n-1} 2^{-k/(n-1)}. \tag{4}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we $\delta$ - separate the cap via lemma 2 we get that the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube $T_\omega$ in the intersection there exists a tube $T_{\omega'}$, such that the angle the angle between $T_\omega$ and $T_{\omega'}$ is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (4) is valid. ☐

## 5. A proof of the Kakeya tube conjecture

We consider a maximal $\delta$-separated set of middle line intersecting tubes $E \subset K_\delta$, where $K_\delta$ is a $\delta$-neighbourhood of a Kakeya set. We can assume that each central line intersects only once by rotating the invidual tubes if necessary a small amount with respect to an axel going through an central line intersection point. This does not affect too much to the volume of the intersections. That is, for any $\delta > 0$ there exists a constant $c_\delta > 0$ s.t after rotations $|\phi_i| < c_\delta$

$$|E''_{2^k}| \sim |E_{2^k}|,$$

where the dyadic set of rotated tubes is defined as

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{\omega' \in \Omega'} 1_{T^\delta_{\omega'}}(x) \leq 2^{k+1}\}.$$

Now, there exists dyadic $k$ such that

$$2^k |E_{2^k}| \approx 1 \tag{5}$$

Moreover there exists $\delta_1 \leq \delta$ s.t all the intersections are central line intersections. Now, it follows for some $k'$ that

$$2^k |E'_{2^{k'}}| \approx \delta_1^{n-1} \delta^{1-n},$$

and we can assume that $k' = k$. This is because $E'_{2^{k'}} \subset \mathrm{E}_{2^{k'}}$ and

$$\begin{aligned} 2^{k'} |E_{2^{k'}}| &\sim \sum_{\omega \in \Omega} \int_{T^\delta_\omega} 1_{E_{2^{k'}}} \geq \sum_{\omega \in \Omega} \int_{T^{\delta_1}_\omega} \delta^{n-1} \delta_1^{1-n} 1_{E'_{2^{k'}}} \\ &\sim \sum_{\omega \in \Omega} \delta_1^{1-n} \delta^{n-1} |T^{\delta_1}_\omega \cap E_{2^{k'}}| \sim 2^{k'} |E'_{2^{k'}}| \delta_1^{1-n} \delta^{n-1} \approx 1. \end{aligned} \tag{6}$$

So we replace the $\delta$-tubes with $\delta_1$-tubes and obtain for the same $k$ via scaling and using the line intersection theorem 2 and lemma 3 that

$$\delta_1^{n-1} \delta^{1-n} \approx 2^k |E'_{2^k}| \lesssim 2^k |S'_{2^k}| \delta_1^{n-1} 2^{-k/(n-1)} \lesssim 2^{-k/(n-1)} \delta_1^{n-1} \delta^{1-n}. \tag{7}$$

Thus,

$$2^k \approx 1. \tag{8}$$

Now, above (8) and (5) implies that

$$|E_{2^k}| \approx 1. \tag{9}$$

The above (9) implies easily the Minkowski version of the Kakeya set conjecture and the Hausdorff version follows from next section.

## 6. A proof of the Kakeya conjecture

We assume that there exist a Kakeya set $K$ that has dimension strictly less than $n$. Let $\bigcup_{j=1}^\infty B_j$ be a cover of $K$ with balls of diameters less than $1 > r > r_j > 0$. Let $n > n - \alpha > 0$ be such that

$$\sum_{j=1}^\infty r_j^{n-\alpha} < 1. \tag{10}$$

If the $n-\alpha-$ dimensional Hausdorff content is zero that kind of cover exists. By compactness of the Kakeya set we can take a subcover with diameters such that $1 > r > r_j \geq \delta > 0$, where at least one $r_j = \delta$. Now, we have proved that

$$\sum_{j=1}^{M} r_j^n \gtrsim |\bigcup_{j=1}^{M} B_j| \gtrsim |\bigcup_{i=1}^{N} T_i| \gtrapprox 1. \tag{11}$$

The second inequality above follows because the balls cover the middle lines of the tubes, so there exists a constant such that the second inequality above is valid. Using inequality (10) and (11) we obtain

$$C_{\alpha/k}\delta^{-\alpha/k}\sum_{j=1}^{M} r_j^n > \sum_{j=1}^{M} r_j^{n-\alpha}. \tag{12}$$

Thus,

$$\sum_{j=1}^{M} r_j^n (C_{\alpha/k}\delta^{-\alpha/k} - r_j^{-\alpha}) > 0. \tag{13}$$

It follows that for the average value of a power of diameters it holds that

$$C_{\alpha/k}\delta^{-\alpha/k} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha} \geq \frac{1}{M^{-\alpha}}(\sum_{j=1}^{M} r_j)^{-\alpha}, \tag{14}$$

where we used Jensen‘s inequality. Thus,

$$c_\alpha \frac{1}{M}\sum_{j=1}^{M} r_j > \delta^{1/k}. \tag{15}$$

From above it follows that

$$\frac{(c_\alpha)^n}{M}(\sum_{j=1}^{M} r_j^n) \geq (\frac{c_\alpha}{M})^n (\sum_{j=1}^{M} r_j)^n > \delta^{n/k},$$

where we used Jensen‘s inequality again. Thus, from above and inequality (10)

$$C_\alpha > M\delta^{n/k}.$$

It follows from above that

$$\delta^{-n/k} C_\alpha > M \tag{16}$$

We can do the steps (12), (13) and (14) again for $\epsilon = \alpha/2$ and obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha}. \tag{17}$$

Let $k$ and a small $\delta$ be such that

$$\delta^{-\alpha/3} > C_\alpha \delta^{-n/k}.$$

From above and inequalities (16) and (17) we obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \delta^{\alpha/3}\sum_{j=1}^{M} r_j^{-\alpha} > \delta^{\alpha/3}\delta^{-\alpha} = \delta^{-2/3\alpha}, \tag{18}$$

which is a contradiction when $\delta$ is small enough. Thus, we have proved theorem 1.

## References

[1] J. Bourgain, *Besicovitch Type Maximal Operators and Applications to Fourier Anal- ysis, Geometric and Functional Analysis 1* (1991), 147-187.
[2] J. Bourgain, *Harmonic analysis and combinatorics: How much may they contribute to each other?*,IMU/Amer. Math. Soc. (2000), 13–32
[3] A. Còrdoba, *The Kakeya Maximal Function and the Spherical Summation Multipli- ers*, American Journal of Mathematics 99 (1977), 1-22.
[4] R.O. Davies, *Some Remarks on the Kakeya Problem*, Proc. Camb. Phil. Soc. 69 (1971), 417-421.
[5] Z. Dvir, *On the Size of Kakeya Sets in Finite Fields*, J. Amer. Math. Soc. 22 (2009), 1093-1097.
[6] K.J. Falconer *The Geometry of Fractal Sets*, Cambridge University Press (1985).
[7] E.Kroc, *The Kakeya problem*, available at http://ekroc.weebly.com/uploads/2/1/6/3/21633182/mscessay-final.pdf
[8] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Education (1986)
[9] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press (1993)
[10] T. Tao, *Lecture Notes*, available at math.ucla.edu/ tao/254b.1.99s/ (1999)
[11] T. Tao, *The Bochner-Riesz Conjecture Implies the Restriction Conjecture*, Duke Math. J. 96 (1999), 363-375.
[12] T.Tao, *From rotating needles to stability of waves: emerging connections between com- binatorics, analysis, and pde*,Notices Amer. Math. Soc., 48(3),(2001),294–303.
[13] T. Tao *https://terrytao.wordpress.com/2009/05/15/the-two-ends-reduction-for-the-kakeya-maximal-conjecture/*
[14] T. Wolff, *An Improved Bound for Kakeya Type Maximal Functions*, Rev. Mat. Iberoamericana 11 (1995), 651-674.
[15] T. Wolff, *Recent work connected with the Kakeya problem* Prospects in mathematics (Princeton, NJ, 1996), (1999),129–162.

*Email address*: jaspegren@outlook.com