THE KAKEYA SET CONJECTURE IS TRUE

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ABSTRACT. In this article we will prove the Kakeya set conjecture. In addition we will prove that in the usual approach to the Kakeya maximal function conjecture we can assume that the tube-sets are maximal. Third, we build a direct connection between line incidence theorems and Kakeya type conjectures.

1. INTRODUCTION

The Kakeya maximal function conjecture and it's variations have gained considerable interest especially after an influential paper by Bourgain (2). For example, it would follow from the conjecture that the Kakeya sets and the Nikodym sets have necessarily full dimensions (11; 12; 8). However, the Nikodym set conjecture is implied by the Kakeya set conjecture (8; 12). The case $n = 2$ was proved by Davies see (5) and the finite field case by Dvir (6). A Kakeya is a set that contains an unit line in every direction. For surveys see (16; 13; 3). Almost all the necessary preliminaries for this paper can be found for example in (8), (11) and in (14). Define the δ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\omega^\delta(a) = \{x \in \mathbb{R}^n : |(x-a) \cdot \omega| \leq \frac{1}{2}, |proj_{\omega^\perp}(x-a)| \leq \delta\}.$$

Moreover, let $f \in L_{loc}^1(\mathbb{R}^n)$. Define the Kakeya maximal function $f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T_\omega^\delta(a)} \int_{T_\omega^\delta(a)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$\|f_\delta^*\|_p \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1}$$

for all $\epsilon > 0$. Remarkably, a bound of the form (1) follows from a bound of the form

$$\|\sum_{\omega \in \Omega} 1_{T_\omega(a_\omega)}\|_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{2}$$

for all $\epsilon > 0$, and for any set of δ-separated of δ - tubes. See for example (12) or (8). We will prove that we need to consider only the case were the set Ω is maximal. As usual we define that "$A \lesssim_\approx B$ iff for all $\epsilon > 0$ and for all $\delta > 0$, it holds that $A \leq C_\epsilon \delta^{-\epsilon} B$. We will prove that

Theorem 1. *Every Kakeya set has full dimension.*

2. A REDUCTION TO THE CASE WHERE THE TUBE-SETS ARE MAXIMAL

Let Ω' be any set of δ - separated directions. We will prove that

$$|| \sum_{\omega' \in \Omega'} 1_{T'_\omega(a_{\omega'})} ||_{p/(p-1)} \leq || \sum_{\omega \in \Omega} 1_{T_\omega(a_\omega)} ||_{p/(p-1)},$$

where Ω is maximal. We construct the set Ω as follows. Let Ω' be the original direction-set and let $\Omega' \subset \Omega'''$ be maximal. Define

$$\Omega'' := \{\omega'' \in S^{n-1} | \omega'' \in \Omega''' / \{\Omega'\}\}.$$

Moreover, let

$$\Omega := \Omega' \cup \Omega''.$$

Clearly, Ω is maximal. We choose the tubes corresponding to directions in Ω' to have origo as their center of masses. Thus, what we do is that we add tubes to the original tube-set so it becomes maximal. Now, we can estimate:

$$|| \sum_{\omega' \in \Omega'} 1_{T'_\omega(a_{\omega'})} ||_{p/(p-1)} \leq || \sum_{\omega' \in \Omega'} 1_{T'_\omega(a_{\omega'})} + \sum_{\omega'' \in \Omega''} 1_{T''_\omega(0)} ||_{p/(p-1)}$$

$$= || \sum_{\omega \in \Omega} 1_{T_\omega(a_\omega)} ||_{p/(p-1)}.$$

Thus, we need only to consider the cases where the tube sets are maximal.

3. PREVIOUSLY KNOWN RESULTS

We will use the following bound for the pairwise intersections of δ - tubes:

Lemma 1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in (8).

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, dene its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an $\delta-$separated subset of Ω.

Lemma 2. *In the notation just defined*

$$1 \leq N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in (8).

4. A LINE INTERSECTION THEOREM AND A LEMMA

We consider a set of lines L where no two lines belong to a common plane containing say, A. Because the number of lines is finite we can always find such an point A. Moreover we assume that each line intersects only once. We will prove a bound for the number line intersections.

Theorem 2. *Let L be a finite set lines, where each line intersects only once. The number of intersections of order k of n lines is less than $\sim n/k$.*

Proof. Let A be a point that does not belong to any plane containing a line from the set L We form a set of distinct planes S, where each intersection belongs to $\sim k$ unique planes and no plane contains two or more intersections. This is possible because the number of intersections is finite. We can choose a suitable point c_i from a line intersecting p_i in a way that no other intersection point p_j belongs to the plane defined by points A, p_i and c_i. In total we have then less than n planes in S. So the number of of intersections of order k is less than $\sim n/k$. $\qquad\square$

Lemma 3. *It holds that*

$$(3) \qquad |\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1} 2^{-k/(n-1)}.$$

Proof. Let us suppose that $2^k \sim \delta^{-\beta}, 0 < \beta \leq n-1$, and let's suppose that tube $T_{\omega'}$ intersecting $T_\omega \cap E_{2^k}$ has it's direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between T_ω and $T_{\omega'}$ is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 1 the intersection

$$(4) \qquad |\bigcap_{i=1}^{2^k} T_i| \leq |T_\omega \cap T_{\omega'} \cap E_{2^k}| \leq |T_\omega \cap T_{\omega'}| \lesssim \delta^{n-1+\beta/(n-1)} \sim \delta^{n-1} 2^{-k/(n-1)}.$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2 we get that the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube T_ω in the intersection there exists a tube $T_{\omega'}$, such that the angle the angle between T_ω and $T_{\omega'}$ is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (4) is valid. $\qquad\square$

5. A proof of the Kakeya set conjecture

We consider a maximal δ-separated set of middle line intersecting tubes $E \subset K_\delta$, where K_δ is a δ-neighbourhood of a Kakeya set. We can assume that each central line intersects only once by rotating the invidual tubes if necessary a small amount with respect to an axel going through an intersection point. This does not affect too much to the volume of the intersections. Now, there exists dyadic k such that

$$(5) \qquad 2^k |E_{2^k}| \approx 1$$

Moreover there exists $\delta_1 \leq \delta$ s.t all the intersections are central line intersections. Now, it follows for some k' that

$$2^{k'}|E_{2^{k'}}| \approx \delta_1^{n-1} \delta^{1-n},$$

and we can assume that $k = k$. This is because $E'_{2^{k'}} \subset E_{2^{k'}}$ and

$$(6) \qquad \begin{aligned} 2^{k'}|E_{2^{k'}}| &\sim \sum_{\omega \in \Omega} \int_{T_\omega^\delta} 1_{E_{2^{k'}}} \geq \sum_{\omega \in \Omega} \int_{T_\omega^{\delta_1}} \delta^{n-1} \delta_1^{1-n} 1_{E'_{2^{k'}}} \\ &\sim \sum_{\omega \in \Omega} \delta_1^{1-n} \delta^{n-1} |T_\omega^{\delta_1} \cap E_{2^{k'}}| \sim 2^{k'} |E'_{2^{k'}}| \delta_1^{1-n} \delta^{n-1} \approx 1. \end{aligned}$$

So we replace the δ-tubes with δ_1-tubes and obtain for the same k via scaling and using the line intersection theorem 2 and lemma 3 that

$$(7) \qquad \delta_1^{n-1} \delta^{1-n} \approx 2^k |E'_{2^k}| \lesssim 2^k 2^{-k} |S'_k| \delta_1^{n-1} 2^{-k/(n-1)} \lesssim 2^{-k/(n-1)} \delta_1^{n-1} \delta^{1-n}.$$

Thus,

(8) $$2^k \approx 1.$$

Now, above (8) and (5) implies that

(9) $$|E_{2^k}| \approx 1.$$

The above (9) implies easily the Minkowski version of the Kakeya set onjecture and the Hausdorff version follows from (1).

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