THE KAKEYA SET CONJECTURE IS TRUE

J.ASPEGREN

Abstract. In this article we will prove the Kakeya set conjecture. In addition we will prove that in the usual approach to the Kakeya maximal function conjecture we can assume that the tube-sets are maximal. Third, we build a direct connection between line incidence theorems and Kakeya type conjectures.

1. Introduction

The Kakeya maximal function conjecture and it's variations have gained considerable interest especially after an influential paper by Bourgain (2). For example, it would follow from the conjecture that the Kakeya sets and the Nikodym sets have necessarily full dimensions (11; 12; 8). However, the Nikodym set conjecture is implied by the Kakeya set conjecture (8; 12).The case $n = 2$ was proved by Davies see (5) and the finite field case by Dvir (6). A Kakeya is a set that contains an unit line in every direction. For surveys see (16; 13; 3). Almost all the necessary preliminaries for this paper can be found for example in (8), (11) and in (14). Define the δ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T^{\delta}_{\omega}(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot\omega| \leq \frac{1}{2}, |proj_{\omega^{\perp}}(x-a)| \leq \delta\}.$$

Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. Define the Kakeya maximal function $f^*_{\delta} : S^{n-1} \to \mathbb{R}$ via

$$f^*_{\delta}(\omega) = \sup_{a\in\mathbb{R}^n} \frac{1}{T^{\delta}_{\omega}(a)} \int_{T^{\delta}_{\omega}(a)} |f(y)|\mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f^*_{\delta}||_p \leq C_{\epsilon}\delta^{-n/p+1-\epsilon}, \tag{1}$$

for all $\epsilon > 0$. Remarkably, a bound of the form (1) follows from a bound of the form

$$||\sum_{\omega\in\Omega} 1_{T_{\omega}(a_{\omega})}||_{p/(p-1)} \leq C_{\epsilon}\delta^{-n/p+1-\epsilon}, \tag{2}$$

for all $\epsilon > 0$, and for any set of δ-separated of δ - tubes. See for example (12) or (8). We will prove that we need to consider only the case were the set Ω is maximal. As usual we define that "$A \lessapprox B$ iff for all $\epsilon > 0$ and for all $\delta > 0$, it holds that $A \leq C_{\epsilon}\delta^{-\epsilon}B$. We define the Minkowski dimension $Dim_M(K)$ of any bounded set K as follows. Let $\delta > 0$ and let K_{δ} be the $\delta-$ neighbourhood of the set K, then

$$Dim_M(K) = n - \lim_{\delta\to 0} \frac{\log|K_{\delta}|}{\log\delta},$$

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if the limit exists. We will prove that

Theorem 1. *Every Kakeya set has full dimension.*

2. A reduction to the case where the tube-sets are maximal

Let Ω' be any set of δ - separated directions. We will prove that

$$||\sum_{\omega'\in\Omega'} 1_{T'_\omega(a_{\omega'})}||_{p/(p-1)} \leq ||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)},$$

where Ω is maximal. We construct the set Ω as follows. Let Ω' be the original direction-set and let $\Omega' \subset \Omega'''$ be maximal. Define

$$\Omega'' := \{\omega'' \in S^{n-1} | \omega'' \in \Omega'''/\{\Omega'\}\}.$$

Moreover, let

$$\Omega := \Omega' \cup \Omega''.$$

Clearly, Ω is maximal. We choose the tubes corresponding to directions in Ω' to have origo as their center of masses. Thus, what we do is that we add tubes to the original tube-set so it becomes maximal. Now, we can estimate:

$$||\sum_{\omega'\in\Omega'} 1_{T'_\omega(a_{\omega'})}||_{p/(p-1)} \leq ||\sum_{\omega'\in\Omega'} 1_{T'_\omega(a_{\omega'})} + \sum_{\omega''\in\Omega''} 1_{T''_\omega(0)}||_{p/(p-1)}$$
$$= ||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)}.$$

Thus, we need only to consider the cases where the tube sets are maximal.

3. Previously known results

We will use the following bound for the pairwise intersections of δ - tubes:

Lemma 1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in (8).

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, dene its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an $\delta-$separated subset of Ω.

Lemma 2. *In the notation just defined*

$$1 \leq N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in (8).

4. A line intersection theorem

We consider a set of lines L where no two lines belong to a common plane containing say, A, and no line intersects two or more so called strict joints. We call any intersection of such lines a strict joint. We will prove a bound for the number of such line intersections.

Theorem 2. *The number of strict joints $|S_k|$ of order k of n lines is less than n/k.*

Proof. We see this by noticing that A, a strict joint B and some non-intersection point of a line C intersecting a strict joint B defines a plane. No strict joint belongs to 2 or more of those planes by asumption. It's clear from the asumptions that those planes are distinct. To each strict joints there corresponds k unique planes. In total we have n planes. So the number of strict joints is less than n/k. □

5. A proof of the Kakeya setl function conjecture

We consider a maximal δ-separated set of middle line intersecting tubes $E \subset K_\delta$, where K_δ is a δ-neighbourhood of a Kakeya set. We can assume that no two central lines of the tubes belong to a same plane going through origo by rotating the tubes if necessary a small amount. This can only increase the number of intersections of positive volume and does not affect too much to their volumes. So via theorem 2 of the last section, the number of central line intersections is less than $\sim \delta^{1-n}$. Next, we prove a lemma.

Lemma 3. *It holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1}2^{-k/(n-1)}. \tag{3}$$

Proof. Let us suppose that $2^k < \delta^{-\beta}, 0 < \beta \leq n-1$, and let's suppose that tube $T_{\omega'}$ intersecting $T_\omega \cap E_{2^k}$, where E_{2^k} is defined via dyadic decomposition

$$E_{2^k} := \{x \in E | 2^k \leq \sum_{\omega \in \Omega} 1_{T_\omega(x)} \leq 2^{k+1}\},$$

has it's direction outside of a cap of size $\sim \delta^{n-1}2^k$ on the unit sphere. Then the angle between T_ω and $T_{\omega'}$ is greater than $\sim \delta 2^{1/(n-1)}$. Thus by lemma 1 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \leq |T_\omega \cap T_{\omega'} \cap E_{2^k}| \leq |T_\omega \cap T_{\omega'}| \lesssim \delta^{n-1}2^{-k/(n-1)}. \tag{4}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1}2^k$. If we δ - separate the cap via lemma 2 we get that the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube T_ω in the intersection there exists a tube $T_{\omega'}$, such that the angle the angle between T_ω and $T_{\omega'}$ is $\sim \delta 2^{k/(n-1)}$ and the inequality (4) is valid. □

Now, there exists dyadic k such that

$$2^k |E_{2^k}| \approx 1. \tag{5}$$

Moreover there exists $\delta_1 \leq \delta$ s.t all the δ_1 - tube intersections are central line intersections. For some k' it holds that

$$2^{k\prime} |E'_{2^{k'}}| \approx \delta_1^{n-1}\delta^{1-n},$$

and we can assume that

$$k = k'.$$

This follows because the scaled version has fewer intersections so that $E'_{2^{k'}} \subset E_{2^{k'}}$. Thus,

$$2^{k'}|E_{2^{k'}}| \sim 2^{k'} \sum_{\omega\in\Omega} \int_{T_\omega^\delta} 1_{E_{2^{k'}}} \geq 2^{k'} \sum_{\omega\in\Omega} \int_{T_\omega^{\delta_1}} \delta^{n-1}\delta_1^{1-n} 1_{E'_{2^{k'}}} \gtrsim 2^{k'} \sum_{\omega\in\Omega} \delta_1^{1-n}\delta^{n-1}|T_\omega^{\delta_1} \cap E'_{2^{k'}}|$$
$$\sim 2^{k'}|E'_{2^{k'}}|\delta_1^{1-n}\delta^{n-1} \approx 1.$$

So we replace the δ-tubes with δ_1-tubes and obtain that

$$\delta_1^{n-1}\delta^{1-n} \approx 2^k|E'_{2^k}| \lesssim |S'_k|\delta_1^{n-1}2^{-k/(n-1)} \lesssim 2^{-k/(n-1)}\delta_1^{n-1}\delta^{1-n}. \tag{6}$$

Thus,

$$2^k \approx 1. \tag{7}$$

Now, above (7) and (5) implies that

$$|E_{2^k}| \approx 1. \tag{8}$$

The above (8) implies easily the Minkowski version of the Kakeya set onjecture and the Hausdorff version follows from (1).

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Email address: jaspegren@outlook.com