The proof of Goldbach's Conjecture

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Abstract

In this paper, I will describe and demonstrate a new method to prove Goldbach's Conjecture. The idea behind my method is to organize all natural numbers in a binary tree, and to find the connections between the even numbers and the prime numbers by using the characteristics of the tree structure.

1. Introduction

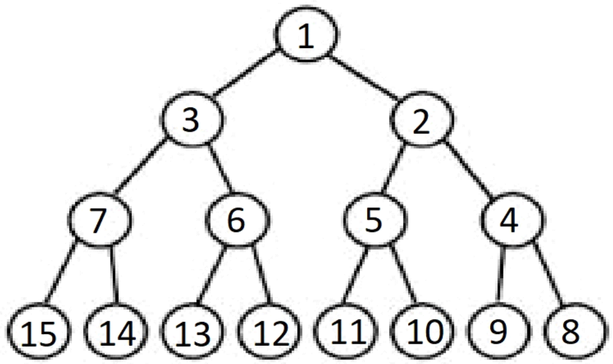
Goldbach's Conjecture states that every even number greater than 2 is the sum of two primes. That is:

$$2n = p_1+p_2, \forall n \in \mathbb{N}$$

and where p1 and p2 depend on n. Since the conjecture was made in 1742, it has not been proven nor been refuted by anyone. Goldbach's conjecture has a weaker version, which states that every odd number greater than 5 can be represented by the sum of three primes. The weaker version was proven in 2013 by **Harald Helgott**. This paper uses the result of Helfgott's proof and introduces a new method to prove Goldbach's Cojecture. The idea behind the method is to organize all natural numbers in a binary tree, and to find the connections between the even numbers and the prime numbers by using the characteristics of the tree structure. Proving Goldbach's Conjecture will improve our knowledge about the frequency of prime numbers, and will be useful for calculations which involve large numbers. Also, it might help with other problems like the ABC conjecture.

2. Definitions and Formulas

Definition 1 The following structure is called a Binary Tree. In some parts of the proof, it is simply called ***tree***.



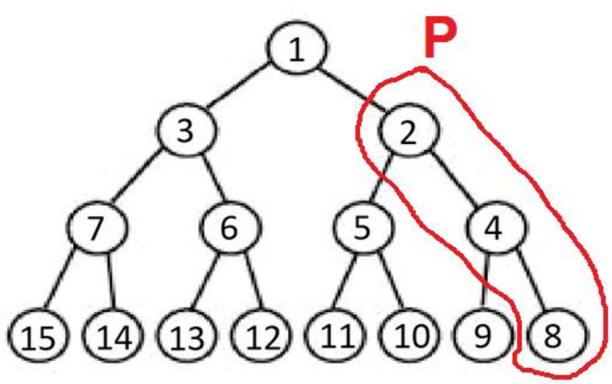
The above tree consists of 4 levels, and contains natural numbers up to 15. Our goal is to proof Goldbach's Cojecture for an infinite amount of even numbers. For this reason, we will assume that the above tree is **infinite** and contains **all natural numbers**.

Definition 2 The symbol V represents an arbitrary vertex in the tree.

Definition 3 $\forall V \in$ tree, the right child of V is an even number and the left child of V is an odd number.

Definition 4 The group **P** contains vertices that are powers of 2.

In mathematical notations: $\mathbf{P} = \{V \in \text{tree} : V = 2^k , k \in \mathbb{N}\}$.

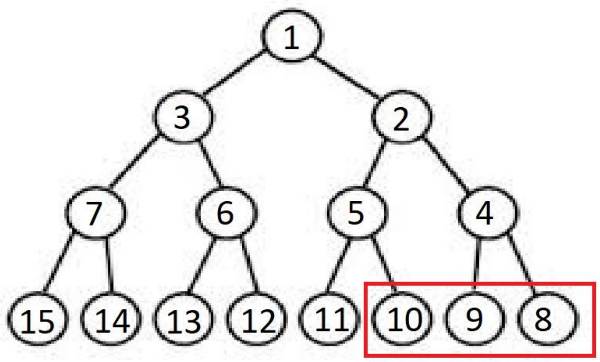


Definition 5 The level k of the tree contains 2^k vertices

Formula 1 $\forall k \in \mathbb{N}\ 2^0+2^1+2^2+....+2^k = \quad 2^{k+1}-1$

Definition 6 The sub-tree which consists of the levels 0,1,2,...k contains $2^0+2^1+2^2+....+2^k$ vertices. According to Formula 1, that is $2^{k+1}-1$ vertices

Definition 7 the **Offset** of V is the number of vertices that exist in the level of V and are located to the right of V. The offset of V always includes V as well.



Definition 8 In every level of the tree, the offsets of even numbers are odd.

Definition 9 In every level of the tree, the offsets of odd numbers are even.

3. The Proof of Goldbach's Conjecture

Theory Goldbach's Cojecture is correct

Proof of the Theory

Lemma 1 Goldbach's Conjecture is correct for $v \in P^c$, while $v = 2m$ and $m \in \mathbb{N}$

Proof of Lemma 1

1. Since the tree is infinite, every even number greater than 2 is in the tree.

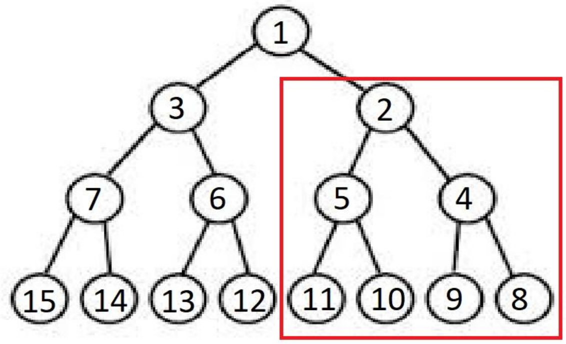
2. Let us choose an arbitrary vertex X in the tree, so that $X = 2m$, $m \in \mathbb{N}$, $m > 1$

3. Let Y be the level of X.

4. Let $R = Y \cap P$.

5. Let T be a sub-tree, with a root that connects X and R together. For example, if $X = 10$

then R = 8 and T will be the following tree:



6. Let A be the offset of vertex X in level Y of the tree T.

7. Let B be the difference between the number of vertices in Y and A, that is: $B = 2^y - A$

Theorem 1.1. Either A is prime or B is prime.

Proof of Theorem 1.1:

1. 2^y is even. According to *definition 8*, A must be odd. Therefore, B must be odd as well.

2. Given that A and B are odd, Let us assume that both A and B are composite numbers. That is, A and B can be divided only by odd numbers. It means that $\{\forall a : A \neq 2^a\}$ and $\{\forall b : B \neq 2^b\}$. Therfore $\{\forall a\ \forall b : A + B \neq 2^a + 2^b\}$, and it follows that $\{\forall a\ \forall b\ \forall y : 2^a + 2^b \neq 2^y\}$. Contradiction!

3. Because of the contradiction in *step 2*, at least one of the two numbers (A and B) must be prime.

4. Q.E.D

8. Let p_1 be a prime number which is equal to either A or B. Let us evaluate the term $X - p_1$.

Theorem 1.2: $X - p_1$ is prime.

Proof of Theorem 1.2:

1. Since X is even and p_1 is prime, the term $X - p_1$ is odd. According to the *Definition 9*, the offset of $X - p_1$ is even. Let 2m be the offset of $X - p_1$.

2. Because of *Definition 6*, $X - p_1 = 2^k - 1 + 2m$

3. Let us assume that X - p_1 is a composite number. If so, it can be divided by an odd number greater than 2. Let's call that number -1+2m

4. It follows that:

$$\frac{(x-p1)}{-1\ +\ 2m} = \frac{2^k-1+2m}{-1+2m} = 1 + \frac{2^k}{-1+2m}$$

5. The number 2^k only has 2 as a prime factor, while the number (-1 + 2m) never has 2 as a prime factor. So 2^k and an (-1 + 2m) will have no prime factors in common. For this reason, the number 2^k cannot be divided by (-1 + 2m).

6. From *step 4* and *step 5*, it follows that the term:

$$\frac{2^k}{-1+2m}$$

is not a natural number, that is, the odd number X - p_1 cannot be divided by an odd number. So X - p_1 is prime.

7. Q.E.D

9. From Theorem 1.1 and Theorem 1.2 it follows that Lemma 1 is proven.

10. Q.E.D

Lemma 2 Goldbach's Conjecture is correct for all $v \in P - \{2\}$

Proof of Lemma 2

Let 2m be an arbitrary even number, so that:

$\{2^x \neq 2m > 2^k : m \in \mathbb{N}, k \in \mathbb{N}, x \in \mathbb{N}, x > k\}$.

It is easy to see that:

$\{\exists n \in \mathbb{N} : 2m = 2^k + 2n\ ,\ 2n \neq 2^x\ \}$

If $2n \neq 2^x$, then $2n \notin P$. Therefore, according to Lemma 1, Goldbach's Conjecture is correct for the number 2n, so $2n = p_1 + p_2$.

It follows that:

$2m = 2^k + p_1 + p_2$

It is known that every odd number is the sum of 3 primes [1].

The number 2m – 3 is an odd number, so it can be expressed as the sum of 3 primes.

Since the number 3 is a prime, 2m is the sum of 4 primes.

That is, $2^k + p_1 + p_2$ is the sum of 4 primes. It follows that 2^k is the sum of 2 primes, thus Goldbach's Conjecture is correct for 2^k $(k \in \mathbb{N}, k > 1)$.

In other words, Goldbach's conjecture is correct for all $v \in P - \{2\}$

Q.E.D

Proposition 1 If Goldbach's Conjecture is correct for all $v \in P^c$, while $v = 2m$ and $m \in \mathbb{N}$ and Goldbach's Conjecture is correct for all $v \in P - \{2\}$, then Goldbach's Conjecture is correct for every even number greater than 2.

Proof of Proposition 1

According to Lemma 1, Goldbach's Conjecture is correct for all $v \in P^c$ while $v = 2m$ and $m \in \mathbb{N}$.

According to Lemma 2, Goldbach's Conjecture is correct for all $v \in P - \{2\}$

It is easy to see that the union of these 2 groups creates a group of all even numbers greater than 2.

Q.E.D.

The proof of Proposition 1 results in the proof of the Theory.

Q.E.D

4. Results

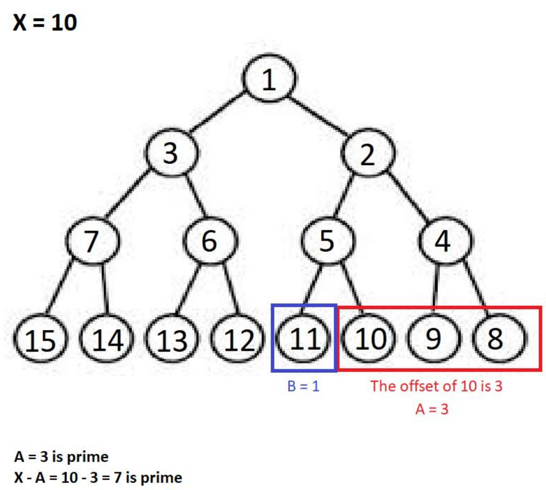
The proof shows that Goldbach’s Conjecture is correct. That is, every even number greater than 2 can be expressed as the sum of two primes.

5. References

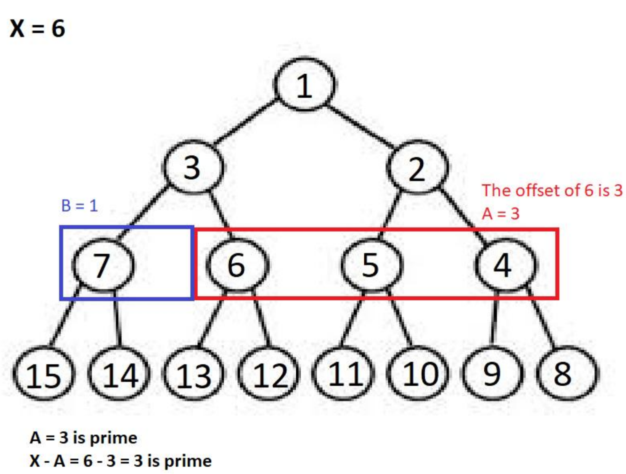
[1] Helfgott, H.A. (2013). "Major arcs for Goldbach's theorem" arXiv:1305.2897

6. Examples:

Is Goldbach's Conjecture correct for the number 10?

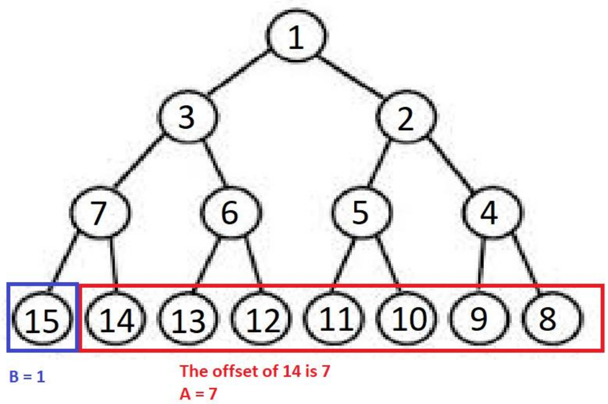


Is Goldbach's Conjecture correct for the number 6?



Is Goldbach's Conjecture correct for the number 14?

X = 14



Is Goldbach's Conjecture correct for the number 12?

A = 12