Question 478 : Some integrals for π^3

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ABSTRACT. This note presents some integrals for π^3 .

1 Introducción

La constante π se define por:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right) = 3.141592\ldots \tag{1}$$

La constante π^3 se define por:

$$\pi^3 = 32\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = 32\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots\right) \tag{2}$$

En esta nota mostramos algunas integrales para la constante π^3 .

Dos fórmulas elementales son:

$$\begin{aligned}\pi^3 &= \frac{81}{5}\int_0^{\infty}\sin^{-1}\left(\sqrt{\frac{3}{3+\left(2e^{\sqrt{x}}-1\right)^2}}\right)dx = \\ &= \frac{81}{5}\int_0^{\infty}\cos^{-1}\left(\frac{2e^{\sqrt{x}}-1}{\sqrt{3+\left(2e^{\sqrt{x}}-1\right)^2}}\right)dx = \frac{81}{5}\int_0^{\infty}\tan^{-1}\left(\frac{\sqrt{3}}{2e^{\sqrt{x}}-1}\right)dx\end{aligned} \tag{3}$$

$$\pi^3 = \frac{81}{5}\int_0^{\pi/3}\left(\ln\left(\frac{1+\sqrt{3}\cot x}{2}\right)\right)^2 dx \tag{4}$$

2 Algunas Integrales para π^3

$$
\begin{aligned}
\pi^3 &= \frac{81}{5}\int_0^\infty \sin^{-1}\left(\sqrt{\frac{3}{3+\left(2e^{\sqrt{x}}-1\right)^2}}\right)dx = \\
&= \frac{162}{5}\int_0^\infty x\sin^{-1}\left(\sqrt{\frac{3}{3+\left(2e^{x}-1\right)^2}}\right)dx = \\
&= \frac{162}{5}\int_1^\infty \frac{\ln x}{x}\sin^{-1}\left(\sqrt{\frac{3}{3+\left(2x-1\right)^2}}\right)dx = \\
&= \frac{162}{5}\int_1^\infty \frac{\ln\left(\left(1+x\right)/2\right)}{1+x}\sin^{-1}\left(\sqrt{\frac{3}{3+x^2}}\right)dx = \\
&= \frac{162\sqrt{3}}{5}\int_{1/\sqrt{3}}^\infty \frac{\ln\left(\left(1+x\sqrt{3}\right)/2\right)}{1+\sqrt{3}x}\sin^{-1}\sqrt{\frac{1}{1+x^2}}\,dx = \\
&= \frac{162\sqrt{3}}{5}\int_0^{\pi/3} \frac{x\ln\left(\left(1+\sqrt{3}\cot x\right)/2\right)}{1+\sqrt{3}\cot x}\sin^2 x\,dx = \\
&= \frac{162}{5}\int_0^1 \frac{-\ln x}{x}\sin^{-1}\left(\frac{x\sqrt{3}}{2\sqrt{1-x+x^2}}\right)dx = \\
&= \frac{162}{5}\int_0^\infty \frac{\ln\left(x+1\right)}{x+1}\sin^{-1}\left(\sqrt{\frac{3}{3+\left(2x+1\right)^2}}\right)dx
\end{aligned} \tag{5}
$$

$$\pi^3 = \frac{81}{5}\int_0^\infty \cos^{-1}\left(\frac{2e^{\sqrt{x}}-1}{\sqrt{3+\left(2e^{\sqrt{x}}-1\right)^2}}\right)dx =$$

$$= \frac{162}{5}\int_0^\infty x\cos^{-1}\left(\frac{2e^{x}-1}{\sqrt{3+\left(2e^{x}-1\right)^2}}\right)dx =$$

$$= \frac{162}{5}\int_1^\infty \frac{\ln x}{x}\cos^{-1}\left(\frac{2x-1}{\sqrt{3+\left(2x-1\right)^2}}\right)dx =$$

$$= \frac{162}{5}\int_1^\infty \frac{\ln\left(\left(1+x\right)/2\right)}{1+x}\cos^{-1}\left(\frac{x}{\sqrt{3+x^2}}\right)dx =$$

$$= \frac{81\sqrt{3}}{5}\int_{\pi/6}^{\pi/2} \frac{\cos^{-1}\left(\sin x\right)}{\cos x\cos\left(x-\left(\pi/3\right)\right)}\ln\left(\frac{1+\sqrt{3}\tan x}{2}\right)dx =$$

$$= \frac{81\sqrt{3}}{10}\int_{\pi/6}^{\pi/2} \frac{\pi-2x}{\cos x\cos\left(x-\left(\pi/3\right)\right)}\ln\left(\frac{1+\sqrt{3}\tan x}{2}\right)dx =$$

$$= \frac{162}{5}\int_0^\infty x\cos^{-1}\left(\frac{2-e^{-x}}{2\sqrt{1-e^{-x}+e^{-2x}}}\right)dx =$$

$$= \frac{162}{5}\int_0^\infty \frac{\ln\left(x+1\right)}{x+1}\cos^{-1}\left(\frac{2x+1}{\sqrt{3+\left(2x+1\right)^2}}\right)dx = \qquad (6)$$

$$= \frac{162}{5}\int_0^1 \frac{-\ln x}{x}\cos^{-1}\left(\frac{2-x}{2\sqrt{1-x+x^2}}\right)dx$$

$$\begin{aligned}
\pi^3 &= \frac{81}{5}\int_0^\infty \tan^{-1}\left(\frac{\sqrt{3}}{2e^{\sqrt{x}}-1}\right)dx = \\
&= \frac{162}{5}\int_0^\infty x\tan^{-1}\left(\frac{\sqrt{3}}{2e^{x}-1}\right)dx = \\
&= \frac{162}{5}\int_1^\infty \frac{\ln x}{x}\tan^{-1}\left(\frac{\sqrt{3}}{2x-1}\right)dx = \\
&= \frac{162}{5}\int_1^\infty \frac{\ln\left((1+x)/2\right)}{1+x}\tan^{-1}\left(\frac{\sqrt{3}}{x}\right)dx = \\
&= \frac{162\sqrt{3}}{5}\int_0^{\sqrt{3}} \frac{\tan^{-1}x}{x\left(x+\sqrt{3}\right)}\ln\left(\frac{1}{2}+\frac{\sqrt{3}}{2x}\right)dx = \\
&= \frac{162\sqrt{3}}{10}\int_0^{\pi/3} \frac{x}{\sin x\sin\left(x+(\pi/3)\right)}\ln\left(\frac{1+\sqrt{3}\cot x}{2}\right)dx = \\
&= \frac{162}{5}\int_0^1 \frac{-\ln x}{x}\tan^{-1}\left(\frac{\sqrt{3}\,x}{2-x}\right)dx = \\
&= \frac{162}{5}\int_0^\infty \frac{\ln(x+1)}{x+1}\tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)dx
\end{aligned} \tag{7}$$

Referencias

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