Refutation of the second incompleteness theorem by Gödel logic

Abstract: Gödel's second incompleteness theorem as based on the minimal modal logic to express the Löb axiom is *not* tautologous. Subsequent substitutions into the Löb axiom along with Hájek's earlier lemma raise further suspicion about Gödel-justification logic.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated *proof* value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). Results are a 16-valued truth table in row-major and horizontal, or repeating fragments of 128-tables for more variables.

LET p, q, r, s: φ or i or x, ψ or y, R, s or z;
~ Not, ¬; + Or, ∨; & And, ∧; = Equivalent, ↔; > Imply, →, greater than;
# necessity, for all or every, □, ∀; % possibility, for one or some, ◊, ∃;
(s=s) ⊤, Tautology as the designated *proof* value; (s@s) ⊥, **F** as contradiction.

From: Holliday, W.H.; Litak, T. (2018). Complete additivity and modal incompleteness. arxiv.org/pdf/1809.07542.pdf

Let vB be the smallest normal modal logic containing the axiom

□◊⊤→ □(□(□p → p) → p), (2.0.1.1)

#%(p=p)>#(#(#p>p)>p) ; **F**N**F**N **F**N**F**N **F**N**F**N **F**N**F**N (2.0.1.2)

which we will call the vB-axiom. Van Benthem [1979] proved that the logic vB is Kripke incomplete.

In this connection, it is noteworthy that the vB-axiom is a theorem of the provability logic GL, the smallest normal modal logic containing the Löb axiom,

□(□p → p) → □p. (2.0.2.1)

#(#p>p)>#p ; CTCT CTCT CTCT CTCT (2.0.2.2)

Remark 2.0.2.: Eq. 2.0.2.2 as rendered is *not* tautologous. This means the Löb axiom is refuted. (From our other papers, the likely intention of the Löb axiom is: □(□p→p)↔(p∨¬p); with a simpler version as either □(□¬p→p)↔□p or □(□p→¬p)↔□¬p.)

Substituting ⊥ for p in the Löb axiom yields

□◊⊤→ □⊥, (2.0.3.1)

#%(p=p)>#(p@p) ; **FFFF FFFF FFFF FFFF** (2.0.3.2) which in the context of provability logic is a modal version of Gödel's Second Incompleteness Theorem.

> **Remark 2.0.3.2**: Eq. 2.0.3.2 is *not* tautologous, meaning in this modal context Gödel's second incompleteness theorem is refuted.

Clearly the vB-axiom is derivable from □◊⊤→ □⊥.

We write this to mean that Eqs. 2.0.3.1 implies 2.0.1.1. (2.0.4.1)

(#%(p=p)>#(p@p)) > (#%(p=p)>#(#(#p>p)>p)) ;
TTTT TTTT TTTT TTTT (2.0.4.2)

In the other direction, van Benthem showed that □◊⊤→ □⊥is a Kripke-frame consequence of the vB-axiom. However, he also showed that □◊⊤→ □⊥is not a theorem of vB.

We write this to mean that Eqs. 2.0.1.1 does not imply 2.0.3.1. (2.0.5.1)

~((#%(p=p)>#(#(#p>p)>p))>(#%(p=p)>#(p@p))) = (p=p) ;
FN**F**N **F**N**F**N **F**N**F**N **F**N**F**N (2.0.5.2)

Together these facts imply the Kripke-incompleteness of vB. (2.0.6.0)

We write this to mean that Eqs. 2.0.4.1 and 2.0.5.1 imply Kripke-incompleteness as defined in Fn. 7 per the Henkin sentence, as "a simplest possible Kripke incomplete unimodal logic" of □(□p↔p)→□p (with the same result table for Eq. 2.0.2.1, the Löb axiom, as *not* tautologous). (2.0.6.1)

(((#%(p=p)>#(p@p))>(#%(p=p)>#(#(#p>p)>p)))&
~((#%(p=p)>#(#(#p>p)>p))>(#%(p=p)>#(p@p))))>
(#(#p=p)>#p) ; TTTT TTTT TTTT TTTT (2.0.6.2)

> **Remark 2.0.6.2:** Eq. 2.0.6.2 has the canonical form of True And False Implies False (TTTT & **F**N**F**N = **F**N**F**N) > CTCT as a theorem. This means that Eq. 2.0.6.0 uses the *non* fact of Eq. 2.0.5.1 to imply the *non* fact of Kripke-incompleteness of vB which is probably not the author intention.

SO[second-order] (□◊⊤→ □⊥), which is equivalent to

$\forall x(\forall y(Rxy \rightarrow \exists zRyz) \rightarrow \forall y\neg Rxy)$, (8.1.1)

(#%(s=s)>#(s@s))=
(((r&(#p&#q))>(r&(#q&s)))>(#q&(~r&(#p&#q))))
TTTC TTTC TTTC TTTT (8.1.2)

by a formalized version of the proof of Lemma 2.1.

Remark 8.1.2: Eq. 8.1.2 is *not* tautologous, meaning the equivalence of Eq. 8.1.1 is denied.

Proposition 9.2 ... [D]erive $\Box\Diamond\top \to \Box\bot$ from the vB-axiom $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$.

We write this as $(\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p))$ implies $(\Box\Diamond\top \to \Box\bot)$. (9.2.1)

((#%(s=s))>#(#(#p>p)>p))>(#%(s=s)>#(s@s)) ;
TCTC TCTC TCTC TCTC (9.2.2)

Remark 9.2.2: Eq. 9.2.2 is *not* tautologous, meaning he derivation is denied.

From: Pischke, N. (2018). A note on strong axiomatization of Gödel-justification logic. arxiv.org/pdf/1809.09608.pdf

Lemma 2.6 (Hájek [*1998*]). G proves the following formulas:

(1) $\varphi \to (\psi \to \varphi)$
(2) $\varphi \to \varphi$
(3) $\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (2.6.3.1)

While item (1) and (2) are even theorems for Hájek's basic logic BL, item (3) is a particular feature of Gödel logic, distinguishing it from the other prominent t-norm based logics. This lemma is also the reason for why the usual proof of the classical deduction theorem works in Gödel logic.

(p>(q>r))>((p>q)>(q>r)) ; T**F**TT TTTT T**F**TT TTTT (2.6.3.2)

Remark 2.6.3.2: Eq. 2.6.3.2 as rendered is *not* tautologous. That particular feature of Gödel logic is refuted along with the lemma as "reason for why the usual proof of the classical deduction theorem works in Gödel logic". Consequently, Gödel-justification logic is suspicious.