Existence of solutions for a nonlinear fractional Langevin equations with multi-point boundary conditions on an unbounded domain

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Abstract

In this work, we apply the fixed point theorems, we study the existence and uniqueness of solutions for Langevin differential equations involving two fractional orders with multi point boundary conditions on the half-line.

Keywords: Riemman-Liouville fractional derivative; fractional Langevin equation; Boundary value conditions; fixed point theorem.

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1 Introduction

...... To be completed.

In this paper, we investigate the existence and uniqueness of solutions for the following fractional Langevin equations with multi point boundary conditions

$$\begin{aligned} &D^{\alpha}\left(D^{\beta}+\lambda\right)y\left(t\right)=f\left(t,y\left(t\right)\right), \qquad 0<t<+\infty, \\ &y(0)=D^{\beta}y\left(0\right)=0, \\ &\lim_{t\to+\infty}D^{\alpha-1}y\left(t\right)=\lim_{t\to+\infty}D^{\alpha+\beta-1}y\left(t\right)=\sum_{i=1}^{m-1}a_i y(\xi_i), \end{aligned} \tag{1}$$

where $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$, such that $1<\alpha+\beta\leqslant 2$, with $a,b\in\mathbb{R}$, $\xi_i\in\mathbb{R}^+, i=1,...,m-1$, and D^{α}, D^{β} are the Riemman-Liouville fractional derivative. By a classical fixed point theorems, several new existence results of solutions are obtained.

2 Preliminaries

Definition 1 *[2] The Riemann-Liouville fractional integral of ordre $\alpha\in\mathbb{R}^+$.for a function $f\in L^1[a,b]$ is defined as*

$$\left(I^{\alpha}f\right)(t)=\frac{1}{\Gamma\left(\alpha\right)}\int_a^t\left(t-\tau\right)^{\alpha-1}f\left(\tau\right)d\tau, \tag{2}$$

where Γ is Gamma Euler function.

Definition 2 *[2] Let $f \in L^1[a,b]$ and $\alpha \in \mathbb{R}^+$ where $n-1<\alpha<n$ with $n \in \mathbb{N}^*$. The Riemann-Liouville direvative of ordre α.for a function f is defined as*

$$D_a^\alpha f(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau. \tag{3}$$

Remark 1 *We have the following properties:*
Let $\delta > 0$ and $\beta > 0$, for all $f \in L^1[a,b]$. Then
i) $I^\delta I^\beta f(t) = I^\beta I^\delta f(t) = I^{\delta+\beta} f(t)$, and $I^\alpha t^\upsilon = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\upsilon}, \quad \upsilon > -1.$
ii) If $\beta > \delta > 0$. Then $D^\delta I^\beta f(t) = I^{\beta-\delta} f(t)$.

Lemma 1 *[2]Let $\alpha \in \mathbb{R}^+$ where $n-1<\alpha \leqslant n$, wiht $n \in \mathbb{N}^*$. Then the differential equation $D^\alpha y(t) = 0$, has this general solution*

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{4}$$

where $c_i \in \mathbb{R}$, with $i = 0,1,2,...,n$.

Lemma 2 *[2]Let $\alpha > 0$. Then*

$$I^\alpha D^\alpha y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{5}$$

where $c_i \in \mathbb{R}$, with $i = 0,1,2,...,n$, and $n-1<\alpha \leqslant n$.

3 Preliminary results

Lemma 3 *Let $h(t) \in C(\mathbb{R}^+, \mathbb{R})$, $1<\alpha \leqslant 2$ and $0<\beta \leqslant 1$, with $1<\alpha+\beta \leqslant 2$. The following problem*

$$\begin{cases} D^\alpha \left(D^\beta + \lambda\right) y(t) = h(t), \qquad t \in (0; +\infty), \\ y(0) = D^\beta y(0) = 0, \\ \lim\limits_{t\to+\infty} D^{\alpha-1} y(t) = \lim\limits_{t\to+\infty} D^{\alpha+\beta-1} y(t) = \sum\limits_{i=1}^{m-1} a_i y(\xi_i), \end{cases} \tag{6}$$

has equivalent to the fractional integral equation

$$\begin{aligned} y(t) \quad = \quad & -\frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds - \mu t^{\alpha+\beta-1} \int_0^{+\infty} h(s) ds \\ & - \frac{\mu\lambda(1+\lambda) t^{\alpha+\beta-1}}{\Gamma(\beta)} \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} y(s)\, ds \\ & + \frac{\mu(1+\lambda) t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha+\beta-1} h(s) ds. \end{aligned} \tag{7}$$

where

$$\mu = \frac{1}{\Gamma(\alpha+\beta) - (1+\lambda) \sum\limits_{i=1}^{m-1} a_i \xi_i^{\alpha+\beta-1}}. \tag{8}$$

Proof. We applied the operator I^α on $D^\alpha \left(D^\beta + \lambda\right) y(t) = h(t)$, we get

$$\left(D^\beta + \lambda\right) y(t) = I^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{9}$$

where $c_1, c_2 \in \mathbb{R}$,
by the boundary condition $y(0) = 0$ and $D^\beta y(0) = 0$, we get $c_2 = 0$, thus

$$D^\beta y(t) = -\lambda y(t) + I^\alpha h(t) + c_1 t^{\alpha-1}, \tag{10}$$

applied the operator I^β

$$y(t) = -\lambda I^\beta y(t) + I^{\alpha+\beta} h(t) + c_1 I^\beta t^{\alpha-1} + c_3 t^{\beta-1}, \tag{11}$$

where $c_3 \in \mathbb{R}$
by the boundary condition $y(0) = 0$ we have $c_3 = 0$, therefore

$$y(t) = -\lambda I^\beta y(t) + I^{\alpha+\beta} h(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \tag{12}$$

Applied the operator $D^{\alpha+\beta-1}$, we get

$$D^{\alpha+\beta-1} y(t) = -\lambda D^{\alpha+\beta-1} I^\beta y(t) + Ih(t) + c_1 \Gamma(\alpha). \tag{13}$$

We have

$$\begin{aligned} D^{\alpha+\beta-1} I^\beta y(t) &= \frac{d}{dt} I^{1-(\alpha+\beta-1)} I^\beta y(t) \\ &= \frac{d}{dt} I^{2-\alpha} y(t) \\ &= \frac{d}{dt} I^{1-(\alpha-1)} y(t) \\ &= D^{\alpha-1} y(t). \end{aligned} \tag{14}$$

Substituting (14) into (13), we obtain

$$D^{\alpha+\beta-1} y(t) = -\lambda D^{\alpha-1} y(t) + Ih(t) + c_1 \Gamma(\alpha), \tag{15}$$

which yields

$$\lim_{t\to+\infty} D^{\alpha+\beta-1} y(t) = -\lambda \lim_{t\to+\infty} D^{\alpha-1} y(t) + \lim_{t\to+\infty} Ih(t) + c_1 \Gamma(\alpha). \tag{16}$$

Using the boundary conditions $\lim\limits_{t\to+\infty} D^{\alpha-1} y(t) = \lim\limits_{t\to+\infty} D^{\alpha+\beta-1} y(t) = \sum\limits_{i=1}^{m-1} a_i y(\xi_i)$ and Eq. (12) we get

$$c_1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \left(-\mu\lambda(1+\lambda) \sum_{i=1}^{m-1} a_i I^\beta y(\xi_i) + \mu(1+\lambda) \sum_{i=1}^{m-1} a_i I^{\alpha+\beta} h(\xi_i) - \mu \lim_{t\to+\infty} Ih(t) \right). \tag{17}$$

where μ defined as in (8).
Substituting the value of c_1 in (12), we get

$$\begin{aligned} y(t) &= -\lambda I^\beta y(t) + I^{\alpha+\beta} h(t) - \mu\lambda(1+\lambda) t^{\alpha+\beta-1} \sum_{i=1}^{m-1} a_i I^\beta y(\xi_i) \\ &\quad + \mu(1+\lambda) t^{\alpha+\beta-1} \sum_{i=1}^{m-1} a_i I^{\alpha+\beta} h(\xi_i) - \mu t^{\alpha+\beta-1} \lim_{t\to+\infty} Ih(t). \end{aligned} \tag{18}$$

Therefore

$$
\begin{aligned}
y(t) \quad = \quad & -\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\, y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\, h(s)ds \\
& -\frac{\mu\lambda(1+\lambda)t^{\alpha+\beta-1}}{\Gamma(\beta)}\sum_{i=1}^{m-1} a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1}\, y(s)\, ds \\
& +\frac{\mu(1+\lambda)t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\sum_{i=1}^{m-1} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha+\beta-1}\, h(s)ds \\
& -\mu t^{\alpha+\beta-1}\int_0^{+\infty} h(s)ds. \qquad (19)
\end{aligned}
$$

The proof is complete ■

Consider the space defined by

$$B = \left\{ y \in C(\mathbb{R}^+, \mathbb{R}),\ \sup_{t\geqslant 0} \frac{|y(t)|}{1+t^{\beta+\alpha-1}} \text{ exist} \right\}, \qquad (20)$$

and with the norm

$$\|y\|_B = \sup_{t\geqslant 0} \frac{|y(t)|}{1+t^{\beta+\alpha-1}}. \qquad (21)$$

Lemma 4 *[1] The space* $(B, \|.\|_B)$ *is Banach space.*

We define the operator $T: E \to E$ by

$$
\begin{aligned}
Ty(t) \quad = \quad & -\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\, y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\, f(s,y(s))ds \\
& -\mu t^{\alpha+\beta-1}\int_0^{+\infty} f(s,y(s))ds - \frac{\mu\lambda(1+\lambda)t^{\alpha+\beta-1}}{\Gamma(\beta)}\sum_{i=1}^{m-1} a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1}\, y(s)\, ds \\
& +\frac{\mu(1+\lambda)t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\sum_{i=1}^{m-1} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha+\beta-1}\, f(s,y(s))ds, \qquad (22)
\end{aligned}
$$

where μ is given by (8).
To be completed.

References

[1] Xinwei Su, Shuqin Zhang, Unbounded solutions to a boundary value problem of fractional order on the half-line,Computers and Mathematics with Applications 61, 1079–1087, (2011).

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204, Elsevier Science B.V, Amsterdam, (2006).

[3] To be completed.