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THE ISOTROPIC CONSTANT CONJECTURE IS TRUE

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Abstract. In this preprint we will prove the isotropic constant conjecture.

1. Introduction

All the problems that we investigate in this preprint have their roots in classical mechanics. However, our tools are from the asymptotic convex geometry. We will prove the following theorem.

Theorem 1. *Let the convex body $D \subset \mathbf{R}^n$ be isotropic. Then*

$$\int_D ||x||^2 dx < nC,$$

where $C > 0$ and C does not depend on n.

Above result implies the hyperplane conjecture [1, 3]. Thus, any convex body of volume 1 has a $n-1$ dimensional hyperplane section that has volume greater than some universal constant [2].

2. Definitions and known results

A convex body $|D| = 1$ is isotropic if the barycenter is at the origin and

$$\int_D < y, x >^2 dx = \alpha^2 ||y||^2,$$

for some constant $\alpha > 0$ and all $y \in \mathbb{R}^n$. The following integral can be calculated in spherical coordinates.

$$\int_{B_n} ||x||^2 dx = \frac{n}{n+2} |B_n|, \tag{1}$$

where B_n is the unit ball.

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3. The proof of the main theorem

Let D be an isotropic convex body. Because centralized balls are isotropic then via scaling there exists a ball L with the same isotropic constant α. So the real question is the volume of that ball L. Now, we integrate the following in spherical coordinates and obtain

$$\frac{1}{|L|^{(n+2)/n}}\int_D ||x||^2dx = \frac{1}{|L|^{(n+2)/n}}\int_L ||x||^2dx = \frac{1}{|L|^{(n+2)/n}}\frac{nR^{n+2}}{n+2}|B_n| = \frac{n}{n+2}|B|^{-2/n}, \tag{2}$$

where B_n is the n dimensional unit ball. Rewriting the above gives

$$\frac{1}{|L|}\int_L ||x||^2dx = \frac{n}{n+2}|B_n|^{-2/n}|L|^{2/n} = c_n|B_n|^{-2/n}, \tag{3}$$

where $c_n > 0$ and

$$\lim_{n\to\infty} c_n = 1.$$

The above limit follows for any family of balls L_n, where we have for each dimension just one ball, when we consider the bound

$$|L| \sim \int_D < y, \phi >^2 dx \leq \sqrt{n}(\log(n))^2C, \tag{4}$$

where $\phi \in S^n$ is an unit vector. For the bound (4) see for example [3] and [2]. In [3] it is proved that there is an universal constant c_1 s.t

$$|L| \leq c_1 \int_L < y, \phi >^2 dx = c_1 \int_D < y, \phi >^2 dx.$$

Thus, from of the bound (4) in each individual dimension n there exists L_n s.t

$$\sup_{i\in I} |L_i| < |L_n| + f(n) \tag{5}$$

for any strictly monotonic $0 < f(n+1) < f(n)$ s.t

$$\lim_{n\to\infty} f(n) = 0$$

Now, it follows from (3) that

$$\sqrt{\frac{n}{n+2}}R_n = \sqrt{c_n}|B_n|^{-1/n} = \sqrt{c_n}r_n,$$

where r_n is the radius of the ball with the unit volume and R_n is the radius of L_n. Let $\epsilon > 0$ then there exists $n \geq 1$ s.t

$$R_n < r_n + \epsilon.$$

Otherwise for some $0 < \alpha < 1$, it holds that

$$r_n + 2\alpha < R_n = C_n r_n,$$

so that for large n,

$$C_n\alpha < R_n - r_n.$$

Thus,

$$C_n(\alpha - r_n) < -r_n \tag{6}$$

and

$$r_n < C_n(r_N - \alpha),$$

which is a contradiction for large n, because

$$\lim_{n\to\infty} C_n = 1.$$

Thus, for large n

$$|L_n| < \frac{|B_n|}{|B_n|}(\frac{r_n + \epsilon}{r_n})^n < C$$

and asymptotically

$$|L_n| \sim 1.$$

Then we have the 1 via equations (3), (5) and a standard estimate

$$|B_n|^{-2/n} \leq Cn.$$

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References

[1] J.Bourgain,*On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math., Vol. 108, No. 6, (1986), 1467–1476.

[2] A.Giannopoulos, *Notes on isotropic convex bodies*, Warsaw, October 2003.

[3] V.D.Milman and A. Pajor *Isotropic Position and Inertia Ellipsoids and Zonoids of the Unit Nall of a Normed n-Dimensional Space*, GAFA-Seminar 87-88, Springer Lecture Notes in Math., v.1376, 64-104, (1989).

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