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A Topological Transformation of Quantum Dynamics

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Abstract

In this work, we discuss the topological transformation of quantum dynamics by showing the wave dynamics of a quantum particle on different types of topological structures in various dimensions from the fundamental polygons of the corresponding universal covering spaces. This is not the view from different perspectives of an observer who simply uses different coordinate systems to describe the same physical phenomenon but rather possible geometric and topological structures that quantum particles are endowed with when they are identified with differentiable manifolds that are embedded or immersed in Euclidean spaces of higher dimension. We present our discussions in the form of Bohr model in one, two and three dimensions using linear wave equations. In one dimension, the fundamental polygon is an interval and the universal covering space is the straight line and in this case the standing wave on a finite string is transformed into the standing wave on a circle which can be applied into the Bohr model of the hydrogen atom. In two dimensions, the fundamental polygon is a square and the universal covering space is the plane and in this case, the standing wave on the square is transformed into the standing wave on different surfaces that can be formed by gluing opposite sides of the square, which include a 2-sphere, a 2-torus, a Klein bottle and a projective plane. In three dimensions, the fundamental polygon is a cube and the universal covering space is the three-dimensional Euclidean space. It is shown that a 3-torus and the manifold $K\times S^1$ defined as the product of a Klein bottle and a circle can be constructed by gluing opposite faces of a cube. Therefore, in three-dimensions, the standing wave on a cube is transformed into the standing wave on a 3-torus or on the manifold $K\times S^1$. We also suggest that the mathematical degeneracy may play an important role in quantum dynamics and be associated with the concept of wavefunction collapse in quantum mechanics.

Keywords

Quantum Topology, Topological Transformation, Quantum Dynamics, Differentiable Geometry and Topology, Differentiable Manifolds, Schrödinger Wave Mechanics, Bohr Model in One, Two and Three Dimensions, de Broglie Wave-Particle Duality

1. Introductory Summary

In our previous works on spacetime structures of quantum particles, we showed that quantum particles can be endowed with various geometric and topological structures of differentiable manifolds and classified according to the mathematical structures that are determined by the wavefunctions that are used to express the geometrical objects associated with the quantum particles, such as the Gaussian curvature and the Ricci scalar curvature. We also showed that many physical properties associated with quantum particles can be determined only by the topological structures rather than the geometric structures, such as angular momentum, electric charge and magnetic monopole [1] [2]. These physical entities can be classified according to topological invariants of the corresponding homotopy groups. Therefore, quantum dynamics is also related closely to the topological structures of a quantum particle [3]. By viewing quantum particles as differentiable manifolds, we also discussed their motion by extending the isometric transformations in classical physics to the isometric embedding between smooth manifolds [4]. In mathematics and physics, the motion of physical objects in an ambient space can be described by geometric transformations under which the properties of the configuration of the objects remain unchanged, such as isometric transformations that preserve the distance from a configuration space onto itself. In classical dynamics, the motion of solid objects can be described by the Poincaré group, which is the non-abelian Lie group of Minkowski spacetime isometries [5] [6]. If we consider quantum particles as differentiable manifolds, then we will need to extend the description of the dynamics of quantum particles in classical physics as point-particles to the dynamics of particles as differentiable manifolds in an ambient space. Furthermore, being viewed as differentiable manifolds, quantum particles are assumed to possess internal geometrical and topological structures that in turns possess internal symmetries that give rise to intrinsic dynamics. If quantum particles are assumed to remain as stable structures, then their intrinsic dynamics should be described by smooth isometric transformations, which are smooth isometric embeddings into the spatiotemporal manifold. The smooth isometric embeddings of differentiable manifolds can also be viewed as geometric solitons which are formed by a continuous process of materialising spacetime structures rather than the motion of a solid physical object through space with respect to time as described in classical physics. However, even though it seems reasonable to apply smooth isometric embeddings into quantum dynamics in which quantum particles are assumed to possess stable geometric structures, such approach will leave out the role played by the topological structures of the differentiable manifolds associated with quantum particles during a dynamical evolution. We may suggest that there should be some kind of internal mechanism that controls the dynamical evolution of the topological structures of a quantum particle responsible for its physical displacement as a differentiable manifold. Such control theories should be rigorously formulated in terms of conformal embeddings, conformal mappings and immersions in differential geometry and topology. In fact, recent developments have shown that unsmooth isometric embeddings can be performed topologically in the sense that these isometric embeddings can change the shape of a physical object without changing its scale. For example, as will be discussed in details in Section 2, the common 2-torus is normally defined as a dough-nut-shaped surface embedded in three-dimensional Euclidean space R^3. The 2-torus can be constructed from the fundamental square by identifying opposite sides of the square, and the embedding requires the fundamental square to be stretched in the third spatial dimension therefore distorts distances. Even so, it is shown that isometric embeddings of the square torus into the ambient three-dimensional Euclidean space can also be performed by modifying the standard torus using C^1 regularity of isometric embeddings to construct C^1 fractal structures from an infinite sequence of waves of corrugations [7]. In spite of that, as illustrations, in this work we will only discuss the topological transformations of quantum dynamics by showing the wave dynamics of a quantum particle on different types of topological structures in various dimensions from the fundamental polygons of the corresponding universal covering spaces. We present and illustrate our discussions in the form of Bohr model in one, two and three dimensions. It should be emphasised that these should not be regarded as the view from the different perspectives of an observer who simply uses different coordinate systems to describe the same physical phenomenon but possible geometric and topological structures that quantum particles are endowed with.

At the macroscopic scale where physical objects are observable, the shape of a physical object depends on the conditions of the environment to which the object belongs. In general, physical objects can change their shapes and other physical features to imitate their environments during the process of evolutionary adaptations. The evolutionary adaptations can be represented in terms of mathematics as processes of geometric and topological evolutions. If we adopt the concept of self-similarity from the fractal theory then at the microscopic scale within the domain of quantum particles we may assume that quantum particles may also have the ability to alter their endowed geometric and topological characteristics to adapt to the environments which are assumed to be composed of physical fields. These physical fields manifest themselves as forces to determine the mathematical structures of quantum particles. We will assume that the topological structures of their associated differentiable manifolds also play an important role in determining the physical structure of a quantum particle and its quantum dynamics. If quantum particles are formed from mass points by contact forces then they may have the ability to change their topological structures to adapt the topological structures of the physical system in which they are part of. For example, if an electron moves in a straight line with a constant speed then it will keep its shape as a particle whose particular geometric and topological structures are stable. However, if it is forced to move in a circle, like moving around the nucleus of a hydrogen atom, then it can turn into the shape of a closed string which vibrates as a standing wave. At each moment of time, the spatial shape of the vibrating string forms a one-dimensional differentiable manifold. Therefore, the geometric and topological processes of evolutionary adaption of quantum particles will determine whether the dynamics is a classical or quantum dynamics. This can be described mathematically as follows. In classical dynamics, the motion of a particle with constant speed in a straight line and the motion of an identical particle with constant speed in a circle are two different dynamical processes that are formulated differently using Newton's second law of motion $m\,\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2 = \boldsymbol{F}$. For the particle that moves in a straight line with constant speed v the acceleration is equal to zero therefore the external net force acting on it is equal to zero. In this case the position x along the straight line is described as $x = vt + x_0$. On the other hand, for the particle which moves in a circle with constant speed v the acceleration a is nonzero and is related to the constant speed v of the motion as $a = v^2/r$. However, these two seemingly distinctive classical dynamics are in fact the same for the case of the electron moving around the nucleus of the Bohr model of a hydrogen atom. To the electron, moving in a circle is also an inertial motion as in the case of moving in a straight line as long as the speed is constant. This problem of dual character of classical and quantum dynamics is probably due to the fact that quantum particles may possess internal geometric and topological structures which may be identified with those of differentiable manifolds. It could be possible that physical laws obeyed by quantum particles are related more closely to the topological structures of a physical system. For example, as will discussed in details later on, even though geometrically the wave dynamics of a quantum particle in a circle is distinctively different from that in a straight line, but topologically they are equivalent because a circle is formed from the fundamental interval of the straight line, which is the universal covering space of the circle. However, in order to give a clearer picture of different geometrical and topological methods that are used to formulate physical laws in physics we now give examples that show how different identifications of physical entities to geometrical objects can lead to different formulations of physical descriptions of the dynamics of classical and quantum mechanics. In physics, classical dynamics describes the motion of physical objects at the macroscopic scale in which the state of motion of an object is determined by the equation of motion which can be derived from the principle of least action. For example, consider a particle moving in a plane under the influence of a force. The normal acceleration a_n of the particle can be found as $a_n = -v^2/\rho$, where v is the speed of the particle and ρ is the radius of curvature. This result can also be obtained by using the variational principle $\delta S = 0$, where S is defined by $S = \int p\mathrm{d}s = \int p\sqrt{1+(\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$, with p is the momentum of the particle [8]. This result not only reveals an intrinsic relationship between geometrical methods and the variational principle in classical mechanics but also reaffirms the belief that the principle of least action can also be used to formulate the physical laws in a deterministic manner. However, we showed that this is not the case when the principle of least action is extended into the domain of quantum mechanics. We showed that the identification of the momentum p of a quantum particle with the de Broglie wavelength λ, which in turns is identified with the curvature κ of the path of a particle, *i.e.* $\kappa = 2\pi/\lambda$, leads to an interesting feature; namely the action principle $\delta S = 0$ is satisfied not only by the stationary path corresponding to the classical motion, but also by any path. In this case the Bohr quantum condition possesses a topological character in the sense that the principal quantum number n is identified with the winding number, which is used to represent the fundamental group of paths [9].

The dual character of classical and quantum dynamics of quantum particles that possess internal geometric and topological structures of differentiable manifolds can also be extended to spaces of higher dimension. In Section 4 we will discuss the topological transformation of the two-dimensional wave dynamics in which quantum particles are assumed to be endowed with the geometric and topological structures of differentiable manifolds of closed vibrating surfaces, such as a 2-sphere, a 2-torus, a Klein bottle, or a hemispherical projective plane. These surfaces can be formed from the fundamental squares of the universal covering plane by the process of gluing opposite sides of the square. As shown in our works on the principle of least action [2] [3] [9], we can generalise Feynman's postulate of random path to formulate a quantum theory in which the transition amplitude between states of a quantum mechanical system is a sum over random surfaces, provided the functional P in the action integral $S = \int P\mathrm{d}A$ is taken to be proportional to the Gaussian curvature K of a surface. Consider classes of surfaces which are described by the higher dimensional homotopy groups. As in the case of the fundamental homotopy group of paths, if we choose from among the homotopy class a representative spherical surface, in which case we can write $\oint P\mathrm{d}A = (q_e/4\pi)\oint K\mathrm{d}A = (q_e/4\pi)\oint \mathrm{d}\Omega = nq_e$. Also as in Bohr model of the hydrogen atom, we may consider a quantum process in which a physical entity transits from one surface to another with some radiation-like quantum created in the process. Since this kind of physical process can be considered as a transition from one homotopy class to another, the radiation-like quantum may be the result of a change of the topological structure of the physical system, and so it can be regarded as a topological effect. It is also noted that the action integral $(q_e/4\pi)\oint K\mathrm{d}A$ is identical to Gauss's law in electrodynamics therefore the constant q_e can be identified with the charge of a particle, which represents the topological structure of a physical system and the charge of a physical system must exist in multiples of q_e. Hence, the charge of a physical system may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. We want to mention here that in differential geometry the Gaussian curvature K is related to the Ricci scalar curvature R by the relation $R = 2K$, and it has been shown that the Ricci scalar curvature can be identified with the potential of a physical system, therefore our assumption of the existence of a relationship between the Gaussian curvature and the surface density of a physical quantity can be justified [1]. Furthermore, by extending Feynman's method of sum over random surfaces to the temporal dynamics in which the magnetic monopole can also be considered as a topological structure of the temporal continuum then we can establish a relationship between the electric charge q_e and the magnetic monopole q_m associated with a quantum particle, similar to Dirac relation $\hbar c/q_e q_m = 2$ [10]. Let P_T be a 3-dimensional physical entity which will be identified with the surface density of a magnetic substance, such as the magnetic charge of an elementary particle. We therefore assume that an elementary particle is assigned not only with an electric charge q_e but also a magnetic charge q_m. We further assume that the quantity P_T is proportional to the temporal Gaussian curvature K_T. Now, if we consider a surface action integral of the form $S = \int P_T \mathrm{d}A_T = \int (q_m/2\pi) K_T \mathrm{d}A_T = n_T q_m$ then the constant q_m can be identified with the magnetic charge of a particle. In particular, the magnetic charge q_m represents the topological structure of a physical system must exist in multiples of q_m. Hence, the magnetic charge of a physical system, such as an elementary particle, may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. We now show that it is possible to obtain Dirac relationship between the electric charge q_e and the magnetic charge q_m by considering a spatiotemporal curvature K which is defined as a product of the temporal Gaussian curvature K_T and the spatial Gaussian curvature K_S as $K = K_T \times K_S$. The spatiotemporal submanifold that gives rise to this form of curvature is homeomorphic to $S^2 \times S^2$. If K_T and K_S are independent from each other then we can write $\oint K \mathrm{d}A = \oint K_T \times K_S \mathrm{d}A_T \mathrm{d}A_S = \oint K_T \mathrm{d}A_T \times \oint K_S \mathrm{d}A_S$. If we assume further that $\oint K \mathrm{d}A = k$, where k is an undetermined constant, then we obtain a general relationship between the electric charge q_e and the magnetic charge q_m as $k/q_e q_m = n_S n_T$. In particular, if $n_S = 1$, $n_T = 2$ and $k = \hbar c$, or $n_S = 2$, $n_T = 1$ and $k = \hbar c$, then we recover the relationship obtained by Dirac, $\hbar c/q_e q_m = 2$. We can then extend our discussions into three dimensions even though we also showed that the entire geometric and topological structures of quantum particles are not observable to an observer in the three-dimensional Euclidean space R^3 if they are formulated as three-dimensional differentiable manifolds embedded or immersed in the four-dimensional Euclidean space R^4 [11]. Nonetheless, it is shown that different three-dimensional manifolds, such as a 3-torus and the $K \times S^1$, which is the product of a Klein bottle and a circle, can be formed by gluing the opposite faces of the fundamental polygon, which is a three-dimensional cube [12], therefore, in Section 5, we will discuss the topological transformation of a three-dimensional wave dynamics in which quantum particles are assumed to be endowed with the geometric and topological structures of differentiable manifolds of a 3-torus or the $K \times S^1$.

Probably, the most prominent feature that emerges from formulating quantum physics in terms of differential geometry and topology is the possibility to express geometric and topological structures of quantum particles by using quantum wavefunctions. We showed that in one dimension, the geometric structure of a 1D differentiable manifold that is represented by the curvature κ can be expressed in terms of a wavefunction ψ as $\kappa = \left(1 + \left(\mathrm{d}\psi/\mathrm{d}x\right)^2\right)^{-3/2} \left(\mathrm{d}^2\psi/\mathrm{d}x^2\right)$, and in two dimensions the Ricci scalar curvature R of a 2D differentiable manifold can be expressed in terms of a wavefunction ψ as $R = 2\left(\psi_{11}\psi_{22} - \psi_{12}^2\right)^2 \Big/ \left(1 + \psi_1^2 + \psi_2^2\right)^2$, where $\psi_\mu = \partial\psi/\partial x^\mu$ and $\psi_{\mu\nu} = \partial^2\psi/\partial x^\mu \partial x^\nu$. However, in three dimensions, despite no direct relationship between the Ricci scalar curvature and the wavefunction that describes a manifold has been established, we have shown that such relationship can be constructed through physical identifications rather than from purely geometrical formulations. For example, we showed that the Ricci scalar curvature that describes the geometrical structure of a quantum particle satisfies the three-dimensional diffusion equation $\partial_t R = k\nabla^2 R$. Solutions to the diffusion equation can be found as $R(x, y, z, t) = \left(M \Big/ \left(\sqrt{4\pi kt}\right)^3\right) \mathrm{e}^{-\left(x^2+y^2+z^2\right)/4kt}$, which determines the probabilistic distribution of an amount of geometrical substance M which manifests as observable matter. As shown in appendix 1, if a pseudo-Euclidean metric is defined in the form $\mathrm{d}s^2 = Dc^2\mathrm{d}t^2 - A(x, y, z, t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$, where D is constant, then the quantity $A(x, y, z, t)$ can be determined [1]. However, for quantum particles that are endowed with the geometric and topological structures of differentiable manifolds, it would be more suitable to consider Bianchi manifolds with a metric which has separate scale factors given by the line element $\mathrm{d}s^2 = Dc^2\mathrm{d}t^2 - a_1(t)^2\,\mathrm{d}x^2 - a_2(t)^2\,\mathrm{d}y^2 - a_3(t)^2\,\mathrm{d}z^2$. Spaces with this form of metric are homogeneous but not generally isotropic therefore quantum particles with this metric will change its volume and shape. The deformation is determined by the shear constructed from the scale factors $a_i(t)$ [13]. On the other hand, we have also shown that the Ricci scalar curvature that describes the geometrical structures of a quantum particle can also be constructed from Schrödinger wavefunctions in wave mechanics. As shown in appendix 2, the relationship between the Schrödinger wavefunction ψ and the Ricci scalar curvature R can be established as

$$R = \left(\sum_{\mu=1}^{3}\left(\mathrm{d}x^\mu/\mathrm{d}t\right)^2 - \hbar\left(\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi\left(\mathrm{d}x^\mu/\mathrm{d}t\right)\right)/m\psi\right)\Big/k.$$

The purpose of this work is to discuss the topological transformation of quantum dynamics by showing the wave dynamics of a quantum particle on different types of topological structures in various dimensions from the fundamental polygons of the corresponding universal covering spaces. Therefore, the topological structures of differentiable manifolds that are associated with quantum particles will hold a dominant role in our discussions. This in fact is a common feature of natural existence not only at the microscopic scale of quantum particles that we assume in this work but also at any scale. A more complete formulation of the dynamics of quantum particles would be a wave dynamics on geometries whose mathematical structures can be classified according to the uniformisation theorem and Thurston geometrisation conjecture. In two dimensions, there are three geometries, which are Euclidean E^2, spherical S^2 and hyperbolic H^2. In three dimensions, Thurston geometrisation conjecture states that every closed three-dimensional manifold can be decomposed into submanifolds which can be constructed from eight types of geometric structures, which are spherical geometry S^3, Euclidean geometry E^3, hyperbolic geometry H^3, the geometry of $S^2 \times R$, the geometry of $H^2 \times R$, the geometry of the universal cover of $SL(2,R)$, Nil geometry, and Solv geometry [14] [15]. And a rigorous treatment of the wave dynamics on these geometries would be geometric wave equations on differentiable manifolds, in particular, linear wave equations on Lorentzian manifolds [16]. However, for the purpose of physical illustration, we will follow a modest approach in which we will present our discussions in the form of Bohr model in one, two and three dimensions using linear wave equations. In one dimension, the fundamental polygon is an interval and the universal covering space is the straight line and in this case the standing wave on a finite string is transformed into the standing wave on a circle which can be applied into the Bohr model of the hydrogen atom. The wave dynamics on a circle can also be described in terms of projective geometry. Since a circle is a 1-sphere which is also a 1-torus therefore the Bohr model of the hydrogen atom can also be viewed as a standing wave on a 1-torus. In two dimensions, the fundamental polygon is a square and the universal covering space is the plane and in this case the standing wave on the square is transformed into the standing wave on different surfaces that can be formed by gluing opposite sides of the square, which include a 2-sphere, a 2-torus, a Klein bottle and a projective plane. We will show when the wave dynamics on a projective plane is described in terms of projective geometry then it is identical to the wave dynamics on a 2-sphere. In three dimensions, the fundamental polygon is a cube and the universal covering space is the three-dimensional Euclidean space. It is shown that a 3-torus and the manifold $K \times S^1$ defined as the product of a Klein bottle and a circle can be constructed by gluing opposite faces of a cube therefore in three-dimensions the standing wave on a cube is transformed into the standing wave on a 3-torus or on the manifold $K \times S^1$. We also discuss a transformation of a stationary wave on the fundamental cube into a stationary wave on a 3-sphere despite it still remains unknown whether a 3-sphere can be constructed directly from a cube by gluing its opposite faces. In spite of this uncertainty, however, we speculate that mathematical degeneracy in which an element of a class of objects degenerates into an element of a different but simpler class may play an important role in quantum dynamics. For example, a 2-sphere is a degenerate 2-torus when the axis of revolution passes through the centre of the generating circle. Therefore, it seems reasonable to assume that if an n-torus degenerates into an n-sphere then wavefunctions on an n-torus may also be degenerated into wavefunctions on an n-sphere. Furthermore, since an n-sphere can degenerate itself into a single point, therefore the mathematical degeneracy may be related to the concept of wavefunction collapse in quantum mechanics where the classical observables such as position and momentum can only be obtained from the collapse of the associated wavefunctions for physical measurements. This consideration suggests that quantum particles associated with differentiable manifolds may possess the more stable mathematical structures of an n-torus rather than those of an n-sphere, therefore, also as a brief investigation into different methods of embeddings of differentiable manifolds in Euclidean spaces, in the next section we will examine the geometric and topological structures of the familiar 2-torus and how it can be isometrically embedded in the ambient three-dimensional Euclidean space R^3.

2. On the Geometric and Topological Structures and the Isometric Embeddings of a 2-Torus

In geometry, when a circle revolves about an axis which does not touch the circle in the three-dimensional Euclidean space R^3 then it generates the surface of revolution of a 2-torus, as shown in the following Figure 1.

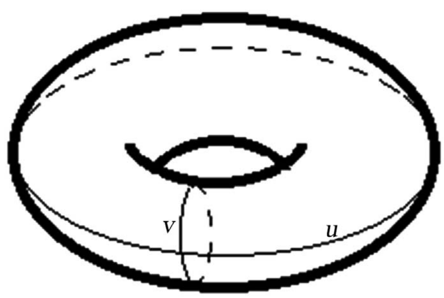

Figure 1. The surface of revolution of a 2-torus.

On the other hand, in topology, a torus can also be defined as the Cartesian product of two circles $S^1 \times S^1$. The homeomorphism between a ring torus and the Cartesian product of two circles leads to an important feature about the embedding of the 2-torus into a higher dimensional Euclidean space. Normally, a 2-torus is simply viewed as a doughnut-shaped surface embedded in three-dimensional Euclidean space R^3. However, if a 2-torus is defined as the Cartesian product $S^1 \times S^1$, called Clifford torus, then since each circle is embedded in a two-dimensional Euclidean space R^2 therefore the product space is a four-dimensional Euclidean space R^4. As a consequence, the embedding of the Clifford torus in four-dimensional Euclidean space R^4 is symmetric and isometric but the embedding of the 2-torus in three-dimensional Euclidean space R^3 is asymmetric and non-isometric [17]. The difference can be specified by using the Gaussian curvature as follows [18]. In the three-dimensional Euclidean space R^3, the parametric equations for a doughnut-shaped torus given in terms of the parameters (u,v) as $x=(c+a\cos v)\cos u$, $y=(c+a\cos v)\sin u$, $z=a\sin v$, where a is the radius of the tube and c is the radius from the centre of the torus to the centre of the tube, and $u,v\in[0,2\pi)$. The line element can then be found as $\mathrm{d}s^2=(c+a\cos v)^2\,\mathrm{d}u^2+a^2\mathrm{d}v^2$. From this line element the Gaussian curvature can be found as $K=\cos v/a(c+a\cos v)$. The ring torus corresponds to $c>a$ for which $K>0$ for the outer region of the torus and $K<0$ for the inner region. On the other hand, the Clifford torus is a flat square torus which is isometric to the fundamental square whose opposite sides are identified as shown below **Figure 2**.

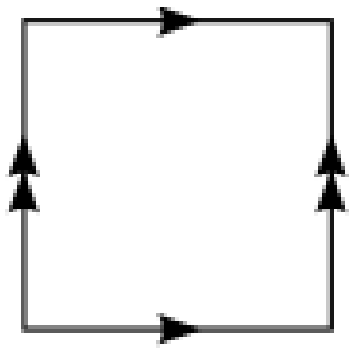

Figure 2. The fundamental square of the Clifford torus.

The isometric embedding of the Clifford torus in the four-dimensional Euclidean space R^4 shows that it is flat and obeys the Euclidean geometry. Then it had emerged the interesting question whether it is possible to isometrically embed the flat 2-torus in three-dimensional Euclidean space R^3. Remarkably, the Nash embedding theorem in topology states that such isometric embedding is possible [19] [20] [21]. It has also been shown that isometric embeddings of the square flat torus into the ambient three-dimensional Euclidean space R^3 can be performed by modifying the standard torus using C^1 regularity of isometric embeddings to construct C^1 fractal structures from an infinite sequence of waves of corrugations. By implementing the Convex Integration Theory, it is possible to visualise isometric embeddings of a flat torus into the ambient three-dimensional Euclidean space R^3. In general, in order to evaluate the curvature at every point of a surface it is required that the surface must be of class C^2. For the case of the flat 2-torus, since the curvature is vanished at every point of the surface therefore it cannot be isometrically embedded with C^2 regularity. However, this does not prevent its isometric embeddings into the three-dimensional Euclidean space R^3 if the embeddings belong to the class C^1, and there are infinitely many such isometric embeddings [7]. Another important development involving fractals that we want to mention here is the study of fractal solutions of linear and nonlinear dispersive partial differential equations on the torus, in particular, fractal solutions of linear and nonlinear Schrödinger wave equations [22]. Fractal images are visual representations of fractal spaces that can also be determined by a system of differential equations that exhibit chaotic dynamics [23] [24]. However, the purpose of this work is to discuss the topological transformation of quantum dynamics of quantum particles therefore in the following we will focus only on linear wave equations on different topological structures that can be formed from the fundamental polygons of their corresponding universal covering spaces in one, two, and three dimensions.

3. Geometric and Topological Transformation of Bohr Model of the Hydrogen Atom

In order to successfully construct a model for the hydrogen atom which predicts correctly the spectrum of the energy radiated from the atom, Bohr proposed three postulates which state that the centripetal force required for the electron to orbit the nucleus in a stable circle is the Coulomb force $mv^2/r = kq^2/r^2$, the permissible orbits are those that satisfy the condition that the angular momentum of the electron equals $n\hbar$, that is $mvr = n\hbar$, and when the electron moves in one of the stable orbits it does not radiate, however, it will radiate when it makes a transition between the stable orbits [25]. On the other hand, in his work on the concept of matter wave, de Broglie proposed that an electron has both a wave and a particle nature by regarding the electron as a standing wave around the circumference of an orbit, as shown in the following **Figure 3** [26].

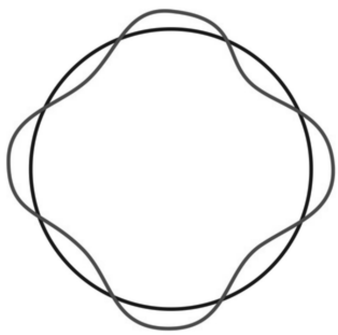

Figure 3. A standing wave around the circumference of a circle.

It is seen that de Broglie's requirement leads to the wave condition $2\pi r = n\lambda$. This is equivalent to assuming that the standing wave around a circle, which is a 1-sphere, is similar to a standing wave on the fundamental interval of a straight line R which is the universal covering space of the circle S^1, where the translations taking the interval to the next images will generate the holonomy group [27]. In mathematics, the circle of radius r is normally considered as a 1-sphere defined by the relation $S^1 = \{x \in R^2 : |x| = r\}$. In fact, the circle is also classified as a 1-torus T^1 which is a topological space equivalent to the quotient space R/Z, therefore, when the fundamental polygon of the universal covering space R is transformed into a circle we actually also transformed it into a 1-torus T^1. Since there is no difference between the topological structures of the 1-sphere S^1 and the 1-torus T^1, the transformation of quantum dynamics from the fundamental polygon into both of them is the same, but as expected, we will show in the next two sections that in higher dimensions this is not the case. Consider a standing wave on a string defined in the domain $D=\{0<x<L\}$ that satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}=0 \tag{1}$$

with the boundary conditions $\psi(0,t)=0$, $\psi(L,t)=0$ and initial conditions $\psi(x,0)=f(x)$, $\partial\psi/\partial t(x,0)=g(x)$. The general solution to the wave equation given in Equation (1) can be found as [28]

$$\psi(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \tag{2}$$

where

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x,\quad B_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\mathrm{d}x \tag{3}$$

Now imagine we convert the finite string into a circle with a radius R where the end points $x=0$ and $x=L$ are joined so that $2\pi R=L$. In order to describe a standing wave on the circle we first consider a two-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}=0 \tag{4}$$

Using the relationship between the polar coordinates (r,θ) and the Cartesian coordinates (x,y) defined by the relations $x=r\cos\theta$, $y=r\sin\theta$, the two-dimensional wave equation given in Equation (4) is rewritten in the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0 \tag{5}$$

Using the method of separation, solutions to the wave equation given in Equation (5) can be expressed in the form $\psi=R(r)\Theta(\theta)T(t)$, then we obtain

$$\frac{1}{c^2}\frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2}-\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}r^2}-\frac{1}{r}\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}r}-\frac{1}{r^2}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}=0 \tag{6}$$

If we consider the wave motion only on the circle of constant radius $r=R$ then the wave equation given in Equation (6) reduces to two separate ordinary differential equations

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2}+c^2\beta^2T=0,\quad \frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}+R^2\beta^2\Theta=0 \tag{7}$$

Solutions to the equations given in Equation (7) can be found as

$$T(t) = A\cos\beta ct + B\sin\beta ct, \quad \Theta(\theta) = C\cos\beta R\theta + B\sin\beta R\theta \tag{8}$$

Using the conditions $\Theta(0)=0$ and $\Theta(2\pi)=0$, we obtain $C=0$ and $\beta = n/2R$. Therefore, $\Theta_n(\theta) = \sin(n\theta/2)$ and the general solutions are given as

$$\psi(\theta,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{nct}{2R} + B_n \sin\frac{nct}{2R}\right)\sin\frac{n\theta}{2} \tag{9}$$

In fact, the resulting wavefunction $\psi(\theta,t)$ in Equation (9) can be obtained directly from Equation (2) by replacing $x = R\theta$ with the condition $2\pi R = L$. It is also interesting to note that the wavefunction $\psi(\theta,t)$ given in Equation (9) can also be considered as a wavefunction on a projective elliptic geometry in which the points of an n-dimensional projective space are identified with the lines that go through the origin of the (n + 1)-dimensional space and are represented by vectors in the (n + 1)-dimensional Euclidean space R^{n+1}. The distance between two points in a projective space can be defined using the metric that specifies the angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ as $d(\boldsymbol{u},\boldsymbol{v}) = \cot^{-1}\left(|\boldsymbol{u}\cdot\boldsymbol{v}|/|\boldsymbol{u}||\boldsymbol{v}|\right)$ [29].

It is also worth mentioning here that the energy spectrum of the Bohr model can be determined if we apply de Broglie wavelength λ defined in terms of the momentum of a quantum particle as $\lambda = h/mv$. Using the wavelength given by the relation $2\pi R = n\lambda$ we obtain $h/mv = 2\pi R/n$, and this leads to the Bohr's postulate of the quantisation of angular momentum $mvR = n\hbar$. Using this relationship and the Coulomb's law $mv^2/R = kq^2/R^2$ then we obtain the expression for the radius of the nth stationary orbit as $R_n = n^2\hbar^2/mkq^2$. Then the energy spectrum E_n can be calculated as follows

$$E_n = T + V = \frac{mv^2}{2} - \frac{kq^2}{R} = -\frac{kq^2}{2R} = -\frac{mk^2q^4}{2\hbar^2 n^2} \tag{10}$$

where R now is the radius of the nth stationary orbit.

4. Geometric and Topological Transformation of a Two-Dimensional Wave Dynamics

In this section we will extend the discussion in Section 3 by considering the transformation of a standing wave on a fundamental square of the universal covering plane R^2 into a standing wave on a 2-dimensional surface which is formed by identifying and gluing the opposite edges of the square. This may be seen as an extension of the Bohr model of the hydrogen atom from one-dimensional manifolds of the 1-sphere and 1-torus embedded in the ambient two-dimensional Euclidean space R^2 into two-dimensional manifolds embedded or immersed in the ambient three-dimensional Euclidean space R^3. As shown in **Figure 4** below, different types of two-dimensional manifolds can be formed by the process of identifying and gluing the opposite pair of the edges a square, including the surfaces of a 2-sphere S^2, a 2-torus T^2, a Klein bottle K^2, and a projective plane P^2 [17].

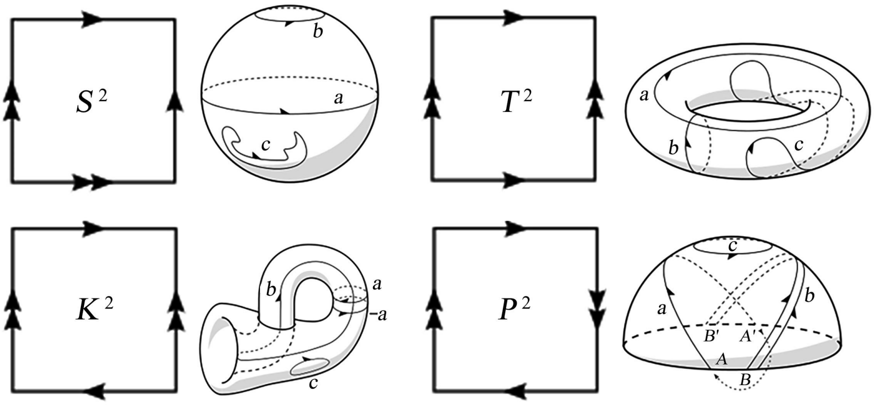

Figure 4. A 2-sphere, a 2-torus, a Klein bottle, a projective plane, and their corresponding fundamental squares.

Each of the four manifolds has a particular fundamental polygon depending on how the edges are identified. However, the universal covering space for all of them is the two-dimensional Euclidean space R^2 where the translations taking the square to the next images will also generate the holonomy group [27]. As in the case of the Bohr model of the hydrogen atom in which the electron is assumed to move in stationary circular orbits, an extended Bohr model on surfaces such as a 2-sphere is similar to Schrödinger wave mechanics in spherical coordinates therefore a moving electron on the surface of a 2-sphere also does not radiate and is described as a standing wave. Therefore we may assume that a standing wave in the universal covering space R^2 can also be transformed into a standing wave on one of the four curved surfaces that can be formed from the fundamental squares as shown above. Consider the standing wave on a square in the domain $D = \{0 < x < L, 0 < y < L\}$ that satisfies the two-dimensional wave equation given in Equation (4) with the boundary conditions $\psi(0, y, t) = 0$, $\psi(L, y, t) = 0$, $\psi(x, 0, t) = 0$, $\psi(x, L, t) = 0$ and initial conditions $\psi(x, y, 0) = f(x, y)$, $\partial\psi/\partial t(x, y, 0) = g(x, y)$. Then the general solution can be found as

$$\psi(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L} \tag{11}$$

where $\lambda_{mn} = (c\pi/L)\sqrt{m^2 + n^2}$, and the coefficients A_{mn} and B_{mn} are given as

$$\begin{aligned} A_{mn} &= \frac{4}{L^2}\int_0^L\int_0^L f(x, y)\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L}\,\mathrm{d}x\mathrm{d}y, \\ B_{mn} &= \frac{4}{L^2\lambda_{mn}}\int_0^L\int_0^L g(x, y)\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L}\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{12}$$

The standing wave described by the function $\psi(x, y, t)$ given in Equation (11) is restricted to the fundamental polygon of the universal covering space R^2. Following the Bohr model of the hydrogen atom in which standing waves on circular orbits are topologically equivalent to standing waves on the fundamental interval of the universal covering line, now we consider the transformation of a standing wave on the fundamental squares into a standing wave on the four corresponding surfaces of a 2-sphere S^2, a 2-torus T^2, a Klein bottle K^2 and a projective plane P^2. In order to describe a standing wave on the transformed surfaces from the fundamental squares, we consider a three-dimensional wave equation given in Cartesian coordinates (x,y,z) of the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}=0 \tag{13}$$

First, consider a standing wave on the surface of a 2-torus. In differential geometry, the relationship between the Cartesian coordinates (x,y,z) and the toroidal coordinates (ξ,η,φ) is given as follows [30] [31]

$$x=\frac{a\sinh\eta\cos\varphi}{\cosh\eta-\cos\xi},\quad y=\frac{a\sinh\eta\sin\varphi}{\cosh\eta-\cos\xi},\quad z=\frac{a\sin\xi}{\cosh\eta-\cos\xi} \tag{14}$$

where the domains of the toroidal coordinates are given as $0\le\xi<2\pi$, $0\le\eta<\infty$, and $0\le\varphi<2\pi$. From the relations given in Equation (14), it can be shown that surfaces of constant $\xi=\xi_0$ correspond to 2-spheres given by the equation $x^2+y^2+(z-a\cot\xi_0)^2=a^2/\sin^2\xi_0$, and surfaces of constant $\eta=\eta_0$ correspond to 2-tori given by the equation $z^2+\left(\sqrt{x^2+y^2}-a\coth\eta_0\right)^2=a^2/\sinh^2\eta_0$. Then in terms of the toroidal coordinates (ξ,η,φ), the three-dimensional wave equation given in Equation (13) can be rewritten as

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{(\cosh\eta-\cos\xi)^3}{a^2\sinh\eta}\left(\frac{\partial}{\partial\xi}\left(\frac{\sinh\eta}{\cosh\eta-\cos\xi}\frac{\partial\psi}{\partial\xi}\right)\right.\\&\left.+\frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\cosh\eta-\cos\xi}\frac{\partial\psi}{\partial\eta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sinh\eta(\cosh\eta-\cos\xi)}\frac{\partial\psi}{\partial\varphi}\right)\right)=0\end{aligned} \tag{15}$$

where $\sinh x$ and $\cosh x$ are hyperbolic functions. Solutions to Equation (15) can be found by separating the variables of the form $\psi(\xi,\eta,\varphi,t)=\sqrt{\cosh\xi-\cos\eta}\,U(\xi)H(\eta)\Phi(\varphi)T(t)$ and then dividing the result by $(\cosh\xi-\cos\eta)^{5/2}U(\xi)H(\eta)\Phi(\varphi)T(t)/\sinh^2\xi$. In this case Equation (15) reduces to the form

$$\begin{aligned}&\frac{\sinh^2\xi}{c^2(\cosh\xi-\cos\eta)^3}\frac{\mathrm{d}^2T}{\mathrm{d}t^2}-\frac{1}{4}\sinh^2\xi-\cosh\xi\sinh\xi\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}\xi}\\&-\sinh^2\xi\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}-\sinh^2\xi\frac{1}{H}\frac{\mathrm{d}^2H}{\mathrm{d}^2\eta}-\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=0\end{aligned} \tag{16}$$

If we consider standing waves only on the surfaces of the toroids which are defined by setting the variable η equal to a constant, $\eta=\eta_0$, then Equation (16) becomes

$$\begin{aligned}&\frac{\sinh^2\xi}{c^2(\cosh\xi-\cos\eta_0)^3}\frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2}-\frac{1}{4}\sinh^2\xi-\cosh\xi\sinh\xi\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}\xi}\\&-\sinh^2\xi\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}-\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=0\end{aligned} \tag{17}$$

By separating the functions in Equation (17), we obtain the following system of ordinary differential equations

$$\frac{\mathrm{d}^2 T}{c^2 \mathrm{d}t^2} + k^2 T = 0 \tag{18}$$

$$\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\varphi^2} + m^2 \Phi = 0 \tag{19}$$

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\xi^2} + \coth\xi \frac{\mathrm{d}U}{\mathrm{d}\xi} - \left(\frac{m^2}{\sinh^2 \xi} - \frac{k^2}{\left(\cosh\xi - \cos\eta_0\right)^3} - \frac{1}{4} \right) U = 0 \tag{20}$$

It is seen from Equations (18-20) that time-independent geometric structures of the extended Bohr model of the hydrogen atom on the surface of a 2-torus can be described by toroidal functions. It is also seen from Equation (18) that if the time dependence of the wave equation is given of the form $\exp(-i\omega t)$, where $\omega = ck$, then the wave equation reduces to the Helmholtz equation $\nabla^2 \psi = \kappa^2 \psi$. It has been shown that solutions to the Helmholtz equation in the toroidal coordinates can be obtained in terms of series representation of the associated Legendre function [32] [33]. It is also noted that a standing wave on the surface of a 2-sphere given by the equation $x^2 + y^2 + \left(z - a\cot\xi_0\right)^2 = a^2 / \sin^2 \xi_0$ can also be obtained from Equation (16) by setting the variable ξ equal to a constant, $\xi = \xi_0$. However, it is more convenient if we follow the common practice using spherical coordinates (r, θ, ϕ) which are related to the Cartesian coordinates (x, y, z) as $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. In spherical coordinates (r, θ, ϕ) the wave equation given in Equation (13) takes the form

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial r^2} - \frac{2}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) - \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = 0 \tag{21}$$

Solutions to Equation (21) can be found by separating the variables of the form $\psi(r, \theta, \phi, t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$. However, if we consider the wave dynamics only on the surface of constant radius $r = R$ then the wavefunction can be written in the form $\psi(\theta, \phi, t) = \Theta(\theta)\Phi(\phi)T(t)$ and the wave equation given in Equation (21) reduces to the following system of ordinary differential equations

$$\frac{1}{c^2}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} + k^2 T = 0 \tag{22}$$

$$\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\phi^2} + m^2 \Phi = 0 \tag{23}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left(k^2 R^2 - \frac{m^2}{\sin^2\theta}\right)\Theta = 0 \tag{24}$$

It should be mentioned here that the 2-sphere S^2 with the constant radius $r = R$ is a spherical membrane which is assumed to vibrate therefore the wavefunction $\psi(R, \theta, \phi, t)$ actually represents the height of the mass points that form the spherical membrane by contact forces. If we let $k^2 R^2 = l(l+1)$ and $\omega = ck$ then general solutions can be found as

$$\psi(\theta,\phi,t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{ml}\cos\omega t+B_{ml}\sin\omega t\right)P_l^m(\cos\theta)\mathrm{e}^{im\phi} \tag{25}$$

The combination $P_l^m(\cos\theta)\mathrm{e}^{im\phi}$ becomes the spherical harmonics $Y_l^m(\theta,\phi)$ when it is normalised. We have shown that a standing wave on the fundamental squares in the universal covering space R^2 can be transformed into a standing wave on a 2-torus or a 2-sphere, respectively. Now, since the 2-dimensional Euclidean space R^2 is also the universal covering space of the Klein bottle K^2 and the projective plane P^2 therefore we can also discuss the possibility to transform a standing wave on their fundamental squares into a standing wave on either of these surfaces. A transformed wave dynamics can be achieved if parametric equations for these two surfaces can be established. For example, the immersion of the Klein bottle in the three-dimensional Euclidean space R^3 is given by the implicit equation $\left(x^2+y^2+z^2+2y-1\right)\left(\left(x^2+y^2+z^2-2y-1\right)^2-8z^2\right)+16xz\left(x^2+y^2+z^2-2y-1\right)=0$, and the parametric equations for the immersion of the Klein bottle are given as

$$x=\left(r+\cos\frac{u}{2}\sin v-\sin\frac{u}{2}\sin 2v\right)\cos u \tag{26}$$

$$y=\left(r+\cos\frac{u}{2}\sin v-\sin\frac{u}{2}\sin 2v\right)\sin u \tag{27}$$

$$z=\sin\frac{u}{2}\sin v+\cos\frac{u}{2}\sin 2v \tag{28}$$

where $u,v\in[0,2\pi)$ and $r>2$. The parameter r is the radius of the self-intersecting circle in the (x,y)-plane, the parameter u gives the angle in the (x,y)-plane, and the parameter v specifies the position of the cross section [34] [35]. Using the parametric equations of the Klein bottle given in Equations (26-28), the wave equation on the fundamental square can be transformed into the wave equation on the surface of the Klein bottle. In general, it is shown in differential geometry that if a metric of the form $\mathrm{d}s^2=g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ can be established on any surface then the Laplacian $\nabla^2\psi$ of a scalar function ψ can be written in the form $\nabla^2\psi=\left(1/\sqrt{g}\right)\partial\left(\sqrt{g}g^{\alpha\beta}\,\partial\psi/\partial x^\beta\right)/\partial x^\alpha$ [36]. Despite the fact that the spherical metric with spherical coordinates (r,θ,ϕ) is obtained from the line element $\mathrm{d}s^2=\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\phi^2$, and the toroidal metric with toroidal coordinates (u,v,θ) is obtained from the line element $\mathrm{d}s^2=\left(a^2/(\cosh u-\cos v)^2\right)\left(\mathrm{d}u^2+\mathrm{d}v^2+\sinh^2 u\mathrm{d}\theta^2\right)$, however, metrics on the Klein bottle and the projective plane are problems that are being investigated. For example, a metric of revolution g_0 for the first eigenvalue on a Klein bottle can be constructed as [37]

$$g_0=\frac{9+\left(1+8\cos^2 v\right)^2}{1+8\cos^2 v}\left(\mathrm{d}u^2+\frac{\mathrm{d}v^2}{1+8\cos^2 v}\right) \tag{29}$$

A more complicated metric on a Klein bottle can also be constructed, for example, in the work from the reference [38]. Even though there is no particular metric that has been established for the surface of the projective plane, we know that it can be represented as the set of all straight lines that pass through the origin and has the structure of a compact surface, as shown in the following Figure 5 [39].

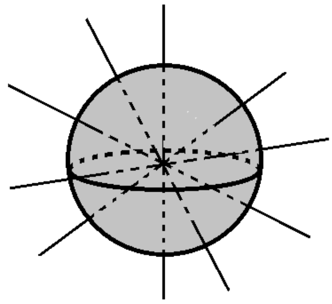

Figure 5. Straight lines representing a projective plane.

Therefore, if the distance between two points in the projective space that immerses in the three-dimensional Euclidean space R^3 can be defined using the metric that specifies the angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ as $d(\boldsymbol{u},\boldsymbol{v}) = \cot^{-1}\left(|\boldsymbol{u}\cdot\boldsymbol{v}|/|\boldsymbol{u}||\boldsymbol{v}|\right)$ then even though they have different geometric natures standing waves on the projective plane can also be described by the solutions given in Equation (25) for standing waves on the surface of a 2-sphere.

5. Geometric and Topological Transformation of a Three-Dimensional Wave Dynamics

In this section we extend further the discussions on the transformations of wave dynamics from a standing wave on fundamental cubes to a standing wave on three-dimensional manifolds that can be formed from the fundamental cubes by the process of gluing opposite surfaces of the cube. This can also be seen as an extension of the Bohr model of the hydrogen atom from a one-dimensional manifold embedded in the ambient two-dimensional Euclidean space R^2 into three-dimensional manifolds embedded or immersed in four-dimensional Euclidean space R^4. As shown in Figure 6 below, a 3-torus can be constructed by identifying the opposite faces of the first cube and the manifold $K \times S^1$, which is the product of a Klein bottle and a circle, can be constructed according to the second cube [12].

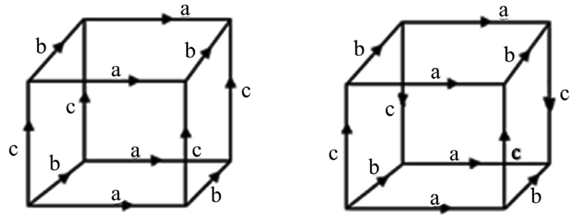

Figure 6. Fundamental cubes of a 3-torus and the manifold which is the product of a Klein bottle and a circle.

Now, consider a standing wave in a cube in a region of three-dimensional Euclidean space defined by the domain $D=\{0<x<L,0<y<L,0<x<L\}$. The standing wave is assumed to satisfy a three-dimensional wave equation written in the Cartesian coordinates (x,y,z) of the form given in Equation (13), with the boundary conditions $\psi(x,y,z,t)=0$ on the boundary of D and the initial conditions $\psi(x,y,z,0)=f(x,y,z)$, $\partial\psi/\partial t(x,y,z,0)=g(x,y,z)$. Then the general solution can be found as

$$\psi(x,y,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\left(A_{lmn}\cos\lambda_{mn}t+B_{lmn}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{L}\sin\frac{m\pi y}{L}\sin\frac{n\pi z}{L} \tag{30}$$

where $\lambda_{mn}=(c\pi/L)\sqrt{l^2+m^2+n^2}$, and the coefficients A_{lmn} and B_{lmn} can be found in terms of the functions $f(x,y,z)$ and $g(x,y,z)$ using the orthogonal conditions. However, the main problem that we are interested in now is how to transform the standing wave in the fundamental cubes into a standing wave on a 3-torus or the manifold $K\times S^1$. In order to describe the wave dynamics on a hypersurface embedded or immersed in four-dimensional Euclidean space R^4 we need a four-dimensional wave equation written in the Cartesian coordinates (x,y,z,w) of the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}-\frac{\partial^2\psi}{\partial w^2}=0 \tag{31}$$

As in the case of two-dimensional surfaces discussed in Section 4, in order to use the wave equation given in Equation (31) to describe a wave dynamics on the surface of a 3-torus we first need to construct a line element $\mathrm{d}s^2=g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ for the 3-torus and then apply the Laplacian $\nabla^2\psi=\left(1/\sqrt{g}\right)\partial\left(\sqrt{g}g^{\alpha\beta}\,\partial\psi/\partial x^\beta\right)/\partial x^\alpha$. The parametric equations for a 3-torus are given as [40]

$$x=(a\cos\theta+b)\cos\varphi,\quad y=(a\cos\theta+b)\sin\varphi,\quad z=a\sin\theta\sin\zeta,\quad w=a\sin\theta\cos\zeta \tag{32}$$

From the parametric equations given in Equation (32), the line element for the 3-torus embedded in the ambient four-dimensional Euclidean space R^4 can be found and given as

$$\mathrm{d}s^2=a^2\left(\mathrm{d}\theta^2+\left(\cos\theta+\frac{b}{a}\right)^2\mathrm{d}\varphi^2+\sin^2\theta\mathrm{d}\zeta^2\right) \tag{33}$$

From the line element given in Equation (33), in terms of the orthogonal coordinates (θ,φ,ζ), the wave equation on the surface of a 3-torus takes the form

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{1}{a^2}\left(\frac{\partial^2\psi}{\partial\theta^2}+\left(-\frac{\sin\theta}{\cos\theta+\frac{b}{a}}+\frac{\cos\theta}{\sin\theta}\right)\frac{\partial\psi}{\partial\theta}\right.\\&\left.+\frac{1}{\left(\cos\theta+\frac{b}{a}\right)^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\zeta^2}\right)=0\end{aligned} \tag{34}$$

It is shown that solutions to the wave equation on 3-torus given in Equation (34) exist and, in particular, they can be written as a Fourier decomposition [41]. For the wave dynamics on surface of the manifold $K \times S^1$, which is the product of a Klein bottle and a circle, we may consider a mixed metric which is a combination of a metric on a circle and a metric on a Klein bottle. A metric on a circle is found using the line element in polar coordiantes $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ in which the radius is constant, $r = R$, therefore, we have $\mathrm{d}s^2 = R^2\mathrm{d}\theta^2$. Together with the metric of revolution g_0 for the first eigenvalue on a Klein bottle given in Equation (29), we obtain

$$\mathrm{d}s^2 = k_1 R^2 \mathrm{d}\theta^2 + k_2\left(\frac{9+\left(1+8\cos^2 v\right)^2}{1+8\cos^2 v}\left(\mathrm{d}u^2 + \frac{\mathrm{d}v^2}{1+8\cos^2 v}\right)\right) \tag{35}$$

where k_1 and k_2 are undetermined constants.

As a further discussion, we now discuss the transformation of a stationary wave on a cube into a stationary wave on a 3-sphere despite it still remains unknown whether a 3-sphere can be constructed directly from a cube by gluing its opposite faces. Despite this uncertainty, however, as mentioned before, we speculate that mathematical degeneracy in which an element of a class of objects degenerates into an element of a different but simpler class may play an important role in quantum dynamics therefore if an n-torus degenerates into an n-sphere then wavefunctions on an n-torus may also be degenerated into wavefunctions on an n-sphere. Consider a d-dimensional hypersphere S_r^d of radius r embedded in the ambient $(d+1)$-dimensional Euclidean space R^{d+1}. If spherical coordinates $(r, \theta, \theta_1, \cdots, \theta_{d-2}, \phi)$ are defined in terms of the Cartesian coordinates $(x_1, x_2, \cdots, x_{d+1})$ as $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\theta_1$, $\cdots$, $x_{d+1} = r\sin\theta \cdots \sin\theta_{d-2}\sin\phi$ then the Laplacian $\nabla^2_{S^d}$ on the hypersphere S_r^d is given as follows [42]

$$\nabla^2_{S^d}\psi = \frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2} + (d-1)\cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\nabla^2_{S^{d-1}}\psi\right) \tag{36}$$

For the case of a 3-sphere S^3 embedded in four-dimensional Euclidean space R^4, the wave equation given in Equation (36) takes the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2} + 2\cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\nabla^2_{S^2}\psi\right) = 0 \tag{37}$$

where $\nabla^2_{S^2}$ is Laplacian operator on a 2-sphere S^2. Solutions to Equation (37) can be established by separating the variables of the form $\psi(r, \theta, \theta_1, \theta_2, t) = R(r)\Theta(\theta, \theta_1, \theta_2)T(t)$. However, if we only consider the wave on the surface of constant radius $r = R$ then the wave equation given in Equation (37) reduces to the following system of ordinary differential equations

$$\frac{1}{c^2}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} + k^2 T = 0 \tag{38}$$

$$\frac{\partial^2\Theta}{\partial\theta^2} + 2\cot\theta\frac{\partial\Theta}{\partial\theta} + \frac{1}{\sin^2\theta}\nabla^2_{S^2}\Theta - k^2R^2\Theta = 0 \tag{39}$$

In particular, for stable quantum particles in which ψ is time-independent therefore we can set $k=0$, and in this case the eigenfunctions of $\nabla^2_{S^3}$ are the hyperspherical harmonics $\Theta^m_{nl}(\theta,\theta_1,\theta_2)$ which are solutions of the equation

$$\nabla^2_{S^3}\Theta^m_{nl}(\theta,\theta_1,\theta_2) = -l(l+2)\Theta^m_{nl}(\theta,\theta_1,\theta_2) \quad (40)$$

It can be shown that $\Theta^m_{nl}(\theta,\theta_1,\theta_2)$ are given as [43]

$$\Theta^m_{nl}(\theta,\theta_1,\theta_2) = 2^{l+1/2}\sqrt{\frac{(n+1)\Gamma(n-l+1)}{\pi\Gamma(n+l+2)}}\Gamma(l+1)\sin^l\theta C^{l+1}_{n-l}(\cos\theta)Y^m_l(\theta_1,\theta_2) \quad (41)$$

where C^{l+1}_{n-l} are the Gegenbauer polynomials and Y^m_l are the 3D spherical harmonics in which $n=0,1,2,\cdots$, $0\le l\le n$, and $-l\le m\le l$. The number of hyperspherical harmonics for a given value of *n* is $(n+1)^2$.

6. Conclusion

In this work, we have discussed the topological transformation of quantum dynamics by showing the wave dynamics of a quantum particle from the fundamental polygons of the corresponding universal covering spaces in one, two and three dimensions. As stated in the introductory summary, this is not the view from different perspectives of an observer who simply uses different coordinate systems to describe the same physical phenomenon but rather possible geometric and topological structures that quantum particles are endowed with when they are identified with differentiable manifolds that are embedded or immersed in Euclidean spaces of higher dimension. For the purpose of physical illustration, we followed a modest approach in which we presented our discussions in the form of Bohr model in one, two and three dimensions using linear wave equations. In one dimension, we considered the topological transformation of a standing wave on a finite string into the standing wave on a circle which can be applied into the Bohr model of the hydrogen atom. The wave dynamics on a circle can also be described in terms of projective elliptic geometry. In two dimensions, we discussed the topological transformation of a standing wave on a square into a standing wave on different surfaces that can be formed by gluing opposite sides of the square, which include a 2-sphere, a 2-torus, a Klein bottle and a projective plane. In particular, we showed that when the wave dynamics on a projective plane is described in terms of projective elliptic geometry, then it is identical to the wave dynamics on a 2-sphere. In three dimensions, we considered the topological transformation of a standing wave on a cube into a standing wave on a 3-torus or on the manifold $K\times S^1$. We also discussed a transformation of a stationary wave on a cube into a stationary wave on a 3-sphere despite it still remains unknown whether a 3-sphere can be constructed directly from a cube by gluing its opposite faces. However, it seems reasonable to assume that if an *n*-torus degenerates into an *n*-sphere, then wavefunctions on an *n*-torus may also be degenerated into wavefunctions on an *n*-sphere. Furthermore, since an *n*-sphere can degenerate itself into a single point, the mathematical degeneracy may be related to the concept of wavefunction collapse in quantum mechanics where the classical observables such as position and momentum can only be obtained from the collapse of the associated wavefunctions for physical measurements. This consideration suggests that quantum particles associated with differentiable manifolds may possess the more stable mathematical structures of an n-torus rather than those of an n-sphere. Even though it has not been discussed in this work, we would like to add the following remark in relation to physical states of quantum particles that would make the topological transformations that have been presented in this work possible. If quantum particles are formed from mass points by contact forces, then they may have the ability to change their topological structures to adapt the topological structures of the physical system in which they are part of. In physics, such ability is related to the physical states of fluid rather than solid states as being assumed for quantum particles in particle physics. And, interestingly, it can be shown that equations of fluid dynamics that describe the fluid state of quantum particles can be derived from Dirac equation in quantum mechanics [44].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendices

Appendix 1

In this appendix, we show in detail the derivation of the equations that are used to determine the metric tensor of the line element given as $ds^2 = Dc^2dt^2 - A(x,y,z,t)(dx^2 + dy^2 + dz^2)$. In differential geometry, the Riemann curvature tensor $R^{\alpha}{}_{\mu\beta\nu}$ is defined in terms of the affine connection $\Gamma^{\gamma}_{\alpha\beta}$ as

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{\partial\Gamma^{\alpha}_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\alpha}_{\mu\nu}}{\partial x^{\beta}} + \Gamma^{\lambda}_{\mu\beta}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\alpha}_{\lambda\beta} \tag{1}$$

The contraction of the Riemann curvature tensor given in Equation (1) with respect to the indices α and β gives the Ricci tensor

$$R_{\mu\nu} = \frac{\partial\Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial\Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} \tag{2}$$

In order to formulate the field equations for the gravitational field it is necessary to introduce a symmetrical metric tensor $g_{\alpha\beta}$ in terms of which the affine connection $\Gamma^{\gamma}_{\alpha\beta}$ is defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \tag{3}$$

With the line $ds^2 = Dc^2dt^2 - A(x,y,z,t)(dx^2 + dy^2 + dz^2)$, we obtain the following non-zero components of the affine connection

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = \frac{1}{2cA}\frac{\partial A}{\partial t},\quad \Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2cA}\frac{\partial A}{\partial t}$$

$$\Gamma^{3}_{03} = \Gamma^{3}_{30} = \frac{1}{2cA}\frac{\partial A}{\partial t},\quad \Gamma^{0}_{11} = \frac{1}{2cD}\frac{\partial A}{\partial t}$$

$$\Gamma^{1}_{11} = \frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^{2}_{11} = -\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^{3}_{11} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{1}_{12} = \Gamma^{1}_{21} = \frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2A}\frac{\partial A}{\partial x},$$

$$\Gamma^{1}_{13} = \Gamma^{1}_{31} = \frac{1}{2A}\frac{\partial A}{\partial z},\quad \Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^{0}_{22} = \frac{1}{2cD}\frac{\partial A}{\partial t},\quad \Gamma^{1}_{22} = \frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^{2}_{22} = \frac{1}{2A}\frac{\partial A}{\partial y}$$

$$\Gamma^{3}_{22} = -\frac{1}{2A}\frac{\partial A}{\partial z},\quad \Gamma^{0}_{33} = \frac{1}{2cD}\frac{\partial A}{\partial t},\quad \Gamma^{1}_{33} = -\frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^{2}_{33} = -\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^{3}_{33} = \frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^{2}_{23} = \Gamma^{2}_{32} = \frac{1}{2A}\frac{\partial A}{\partial z},\quad \Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{1}{2A}\frac{\partial A}{\partial y} \tag{4}$$

From the components of the affine connection given in Equation (4), we obtain

$$R_{11}=\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2}-\frac{1}{A}\frac{\partial^2 A}{\partial x^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial y^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial z^2}+\frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2+\frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{22}=\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial x^2}-\frac{1}{A}\frac{\partial^2 A}{\partial y^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial z^2}+\frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2+\frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{33}=\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial x^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial y^2}-\frac{1}{A}\frac{\partial^2 A}{\partial z^2}+\frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2+\frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{00}=-\frac{3}{2c^2A}\frac{\partial^2 A}{\partial t^2}+\frac{3}{4c^2A^2}\left(\frac{\partial A}{\partial t}\right)^2 \tag{5}$$

Using the relation $R=g^{00}R_{00}+g^{11}R_{11}+g^{22}R_{22}+g^{33}R_{33}$ the Ricci scalar curvature can be found as

$$R=-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\nabla^2 A+\frac{3}{2A^3}(\nabla A)^2 \tag{6}$$

Using the relation

$$R(x,y,z,t)=\left(M\Big/\left(\sqrt{4\pi kt}\right)^3\right)\mathrm{e}^{-\left(x^2+y^2+z^2\right)/4kt} \tag{7}$$

Then we finally arrive at

$$-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\nabla^2 A+\frac{3}{2A^3}(\nabla A)^2=\frac{M}{\left(\sqrt{4\pi kt}\right)^3}\mathrm{e}^{-\frac{x^2+y^2+z^2}{4kt}} \tag{8}$$

Appendix 2

In this appendix, we will show that Schrödinger wavefunctions can be used for the construction of spacetime structures of the quantum states of a quantum system. Schrödinger's original works were on the time-independent quantum states of the hydrogen atom, commencing with the Hamilton-Jacobi equation, written in terms of the Cartesian coordinates (x,y,z) as

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial x}\right)^2-2m\left(E+\frac{kq^2}{r}\right)=0 \tag{1}$$

However, in order to obtain a partial differential equation that would give rise to the required results, Schrödinger introduced a new function ψ, which is real, single-valued and twice differentiable, through the relation $S=\hbar\ln\psi$, where the action S is defined by $S=\int L dt$ and L is the Lagrangian defined by $L=T-\varphi$, with T is the kinetic energy and φ is the potential energy. In terms of the new function ψ, Equation (1) takes the form

$$\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial x}\right)^2-\frac{2m}{\hbar^2}\left(E+\frac{kq^2}{r}\right)\psi^2=0 \quad (2)$$

Then by applying the principle of least action $\delta\int L\mathrm{d}t=0$, Schrödinger arrived at the required equation

$$\nabla^2\psi+\frac{2m}{\hbar^2}\left(E+\frac{kq^2}{r}\right)\psi=0 \quad (3)$$

Now we show that Schrödinger wavefunction ψ can be used to construct the spacetime structures of the quantum states of the hydrogen atom. By using the relations $L=\mathrm{d}S/\mathrm{d}t$, $\mathrm{d}S/\mathrm{d}t=\partial_t S+\sum_{\mu=1}^{3}\partial_\mu S\left(\mathrm{d}x^\mu/\mathrm{d}t\right)$, $T=m\sum_{\mu=1}^{3}\left(\mathrm{d}x^\mu/\mathrm{d}t\right)^2$ and $\varphi=T-L$, we obtain

$$\varphi=m\sum_{\mu=1}^{3}\left(\mathrm{d}x^\mu/\mathrm{d}t\right)^2-\partial_t S+\sum_{\mu=1}^{3}\partial_\mu S\left(\mathrm{d}x^\mu/\mathrm{d}t\right) \quad (4)$$

In terms of the Schrödinger wavefunction ψ, Equation (4) can be rewritten as

$$\varphi=m\sum_{\mu=1}^{3}\left(\mathrm{d}x^\mu/\mathrm{d}t\right)^2-\hbar\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi\left(\mathrm{d}x^\mu/\mathrm{d}t\right)}{\psi} \quad (5)$$

From Poisson equation we can assume the relation $\varphi=kmR$ then the following relation between the Schrödinger wavefunction ψ and the Ricci scalar R can be established [1]

$$R=\frac{1}{k}\left(\sum_{\mu=1}^{3}\left(\mathrm{d}x^\mu/\mathrm{d}t\right)^2-\frac{\hbar}{m}\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi\left(\mathrm{d}x^\mu/\mathrm{d}t\right)}{\psi}\right) \quad (6)$$