Null-cone integral formulation of QED

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Abstract

It is shown that a transformation of the Dirac equation to a zero sum integral over the past null cone, together with a simple transformation of the electromagnetic field source equation, yields a series, each of whose terms corresponds to one Feynman diagram. A feature of this alternative formulation of QED is that neither propagator factors nor off-shell states appear explicitly.

We will denote an infinitesimal element of the past ($t = -r$) null cone by

$$d\Lambda \equiv \delta_+(t^2 - r^2)d^4x = \frac{d^3\mathbf{r}}{t}$$

We will employ the conventions used by Feynman *viz.*
$k^2 \equiv k^\nu k_\nu$
$\not{A} = \gamma^\mu A_\mu, \not{x} = \gamma^\mu x_\mu, \mathbf{I} = \gamma^0\gamma^0$
We will make use of the fact that, for all $k^2 \neq 0$

$$\int_0^\infty e^{ik_\nu x^\nu} d\Lambda = \frac{1}{k^2} \tag{1}$$

and that

$$\int_0^\infty i\not{x}e^{ik_\nu x^\nu} d\Lambda = \frac{\not{k}}{k^4} \tag{2}$$

1 The basic idea

Integration over the null cone can hence be seen as the functional inverse of Lorentz invariant differentiation:

$$\Box \int_0^\infty f(x)d\Lambda = \partial_\mu \Box \int_0^\infty x^\mu f(x)d\Lambda = f(0) \tag{3}$$

(3) implies that solutions $\Psi(x)$ of the Dirac equation for an electron in the presence of an electromagnetic field satisfy:

$$\int_0^\infty [\mathbf{I} + i\not{x}(m + e\not{A})]\Psi(x)d\Lambda = 0 \tag{4}$$

$$A_\nu = \int J_\nu d\Lambda$$

We can define a tensor $\mathbf{\Phi}$ with source equation

$$\Phi_{\mu\nu} = e \int x_\mu J_\nu d\Lambda \tag{5}$$

for which

$$\partial^\nu \Phi_{\mu\nu} = eA_\nu \tag{6}$$

by virtue of (3).
Rewriting (4) in terms of $\mathbf{\Phi} \equiv \gamma^\nu \gamma^\mu \Phi_{\mu\nu}$ yields

$$\int_0^\infty e^{i\mathbf{\Phi}(x)}[\mathbf{I} + im\not{x}]\Psi(x)d\Lambda = 0 \tag{7}$$

This form of the Dirac equation is explicitly gauge invariant since a scalar $e^{i\phi(x)}$ commutes with $[\mathbf{I} + im\not{x}]$ and can be cancelled by applying an opposite phase shift to Ψ.

2 Application to scattering processes

Each term of the infinite series $e^{i\mathbf{\Phi}} = \mathbf{I} + i\mathbf{\Phi} - \frac{1}{2}\mathbf{\Phi}^2 - \frac{i}{6}\mathbf{\Phi}^3 + ...$ corresponds to a different Feynman diagram, so in addition to providing steady state Dirac solutions, (7) yields the scattering amplitude for all single electron diagrams, to all orders in e. Despite this inherent complexity, $e^{i\mathbf{\Phi}}$ is still amenable of being expanded in momentum space *viz*:

$$e^{i\mathbf{\Phi}} = \gamma_\mu \gamma_\nu \int \phi^{\mu\nu}(q) e^{-iq\cdot x} d^4q \tag{8}$$

Given an arbitrary vector potential field

$$\not{A}(x) = \int \not{a}(q) e^{-iq\cdot x} d^4q \tag{9}$$

the first order term by virtue of (6) is:

$$i\mathbf{\Phi} = e \int [\not{q}]^{-1} \not{a}(q) e^{-iq\cdot x} d^4q \tag{10}$$

Given an initial momentum eigenstate for which $[\not{p}_1 - m] |\mathbf{p_1}\rangle = 0$, first order (elastic) scattering processes make the following contribution to the integrand in (7):

$$ie \cdot e^{-iq\cdot x} \left\|\not{p}_1 + \not{q}\right\|^{-4} [\not{q}]^{-1} \not{a}(q)(\not{p}_1 + \not{q})\not{q} |p_1\rangle \tag{11}$$

Satisfaction of (7) requires the addition of contributions from final states of momentum $\not{p}_2 = \not{p}_1 + \not{q}$. Using the boundary condition of a pure initial state of momentum p_1, these contributions must increase linearly from zero at $t = 0$ and are hence of the form $[1 - e^{-i\Omega t}] |p_2\rangle$, where $\Omega(q)$ is the transition rate appearing in the Golden Rule. The contribution to the integrand in (7) due to the fact that these states are slightly ($\Omega \lll m$) off the mass shell is

$$\gamma_0 \Omega(q) m^{-3} |p_2\rangle \tag{12}$$

By virtue of the orthogonality condition:

$$\int_0^\infty \langle f| \gamma_0 \not{x} |i\rangle \, d\Lambda = \delta_{if} \tag{13}$$

..the contributions (11) and (12) will satisfy the zero sum condition in (7) iff

$$\Omega(q) = ie \cdot m^3 \langle p_2| \left\| \not{p}_1 + \not{q} \right\|^{-4} [\not{q}]^{-1} \not{a}(q)(\not{p}_1 + \not{q})\not{q} \, |p_1\rangle \tag{14}$$

In the non-relativistic limit where $p \ll q$

$$\Omega(q) \approx ie \cdot \langle p_2| \, [\not{q}]^{-1} \not{a}(q) \not{p}_1 m^{-1} \not{q} \, |p_1\rangle \tag{15}$$

If the field is time-independent, $q_0 = 0$ and we obtain the well-known result for elastic scattering:

$$\Omega(q) \approx ie \cdot \langle p_2| \, \not{a}(q) \, |p_1\rangle \tag{16}$$

Returning to (9), the second order term is:

$$-\frac{1}{2}\mathbf{\Phi}^2 = -e^2 \frac{1}{2} \left[\int [\not{q}]^{-1} \not{a}(q) e^{-iq\cdot x} d^4q, \int [\not{q}]^{-1} \not{a}(q) e^{-iq\cdot x} d^4q \right] \tag{17}$$

In the special case that the field comprises just two momentum components viz.

$$\not{A}(x) = \not{a}_1 e^{-iq_1 \cdot x} + \not{a}_2 e^{-iq_2 \cdot x} \tag{18}$$

this simplifies to

$$-\frac{1}{2}\mathbf{\Phi}^2 = -e^2 \frac{1}{2} \left[[\not{q}_1]^{-1} \not{a}_1, [\not{q}_2]^{-1} \not{a}_2 \right] e^{-i(q_1+q_2)\cdot x} \tag{19}$$

Following the same procedure as that which we obtained (11) from (7) yields

$$-\frac{e^2}{2} \cdot e^{-i(q_1+q_2)\cdot x} \left\| \not{p}_1 + \not{q}_1 + \not{q}_2 \right\|^{-4} \left[[\not{q}_1]^{-1} \not{a}_1, [\not{q}_2]^{-1} \not{a}_2 \right] (\not{p}_1 + \not{q}_1 + \not{q}_2)(\not{q}_1 + \not{q}_2) \, |p_1\rangle \tag{20}$$

with 2nd order processes such as Compton scattering and Bremsstrahlung having transition rates in to final states of momentum $\not{p}_2 = \not{p}_1 + \not{q}_1 + \not{q}_2$ given by:

$$\Omega(q) \approx \frac{e^2}{2} \langle p_2| \left[[\not{q}_1]^{-1} \not{a}_1, [\not{q}_2]^{-1} \not{a}_2 \right] \not{p}_1 m^{-1} (\not{q}_1 + \not{q}_2) \, |p_1\rangle \tag{21}$$

Higher order processes can be calculated in an analogous manner.