DIRICHLET PROBLEM FOR HERMITIAN-EINSTEIN EQUATIONS OVER BI-HERMITIAN MANIFOLDS

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Abstract. In this paper, we solve the Dirichlet problem for $\alpha$-Hermitian-Einstein equations on $I_{\pm}$-holomorphic bundles over bi-Hermitian manifolds. As a corollary, we obtain an analogue result about generalized holomorphic bundles on generalized Kähler manifolds.

1. Introduction

A bi-Hermitian structure on a $2n$-dimensional manifold $M$ consists of a triple $(g, I_+, I_-)$, where $g$ is a Riemannian metric on $M$ and $I_\pm$ are integrable complex structures on $M$ that are both orthogonal with respect to $g$. Let $(M, g, I_+, I_-)$ be a bi-Hermitian manifold. Let $E$ be a holomorphic vector bundle on $M$ endowed with two holomorphic structures $\partial_+$ and $\partial_-$ with respect to the complex structures $I_+$ and $I_-$, respectively. Suppose $H$ is a Hermitian metric on $E$. Let $F^H_\pm$ be the curvatures of the Chern connections $\nabla^H_\pm$ on $E$ associated to the Hermitian metric $H$ and the holomorphic structures $\partial_\pm$. Motivated by Hitchin [14], Hu et al. [16] introduced the following $\alpha$-Hermitian-Einstein equation, where $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}$:

\[
\sqrt{-1}(\alpha F_+^H \wedge \omega_+^{n-1} + (1 - \alpha) F_-^H \wedge \omega_-^{n-1}) = (n-1)! \lambda \cdot \text{Id}_E \cdot \text{vol}_g,
\]

where $\omega_\pm(\cdot, \cdot) = g(I_\pm \cdot, \cdot)$ are the fundamental 2-forms of $g$. Once $I_+ = I_-$, (1.1) reduces to the Hermitian-Einstein equation. A Hermitian metric $H$ on $E$ is called $\alpha$-Hermitian-Einstein if it satisfies (1.1).

Recently, the existence of Hermitian-Einstein metrics on holomorphic vector bundles has attracted a lot of attention. The celebrated Donaldson-Uhlenbeck-Yau theorem states that holomorphic vector bundles over compact Kähler manifolds admit Hermitian-Einstein metrics if they are polystable. It was proved by Narasimhan and Seshadri [26] for compact Riemann surface, by Donaldson [8] for algebraic manifolds and by Uhlenbeck and Yau [34] for general compact Kähler manifolds. There are many interesting generalized Donaldson-Uhlenbeck-Yau theorem (see the References [1, 2, 3, 4, 13, 15, 16, 17, 18, 19, 20, 21, 25, 27, 36], etc.). It is natural to hope that geometric results dealing with closed manifolds will extend to yield interesting information for manifolds with boundary. In [9], Donaldson solved the Dirichlet problem for Hermitian-Einstein equations over compact Kähler manifolds with non-empty boundary. Zhang [37] generalized Donaldson’s result to the general Hermitian manifolds. Later, Li and Zhang [23] solved the Dirichlet problem for a class of vortex equations, which generalize the well-known Hermitian-Einstein
equations. At the same time, Zhang [38] also solved the the Dirichlet problem for Hermitian Yang-Mills-Higgs equations for holomorphic vector bundles on compact Kähler manifolds.

Just very recently, Hu et al. [16] proved that the $I_{\pm}$-holomorphic vector bundles admit $\alpha$-Hermitian-Einstein metrics if f they are $\alpha$-polystable, for any $\alpha \in (0, 1)$. In this paper, we want to consider the Dirichlet boundary value problem for $\alpha$-Hermitian-Einstein equations. We obtain the following theorem.

**Theorem 1.1.** Let $(M, g, I_+, I_-)$ be a compact bi-Hermitian manifold with non-empty boundary $\partial M$ such that $\text{vol}_g = \omega_{\pm}^n$. Suppose $(E, \overline{\partial}_+, \overline{\partial}_-)$ is an $I_{\pm}$-holomorphic bundle on $M$. Then for any Hermitian metric $\varphi$ on the restriction of $E$ to $\partial M$, there is a unique $\alpha$-Hermitian-Einstein metric $H$ on $E$ such that $H = \varphi$ on $\partial M$.

**Remark 1.2.** In Theorem 1.1, we assume the bi-Hermitian manifold $(M, g, I_+, I_-)$ satisfying $\text{vol}_g = \omega_{\pm}^n$. The existence of such manifold can be found in Remark 6.14 in [10]. In this case, one can rewrite (1.1) as

$$\alpha\sqrt{-1}\Lambda_+ F_+^H + (1 - \alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \text{Id}_E = 0,$$

where $\Lambda_{\pm}$ are the contraction operators associated to $\omega_{\pm}$, respectively.

Our motivation for studying such bundles also comes from generalized complex geometry. In [11], Gualtieri introduced generalized holomorphic bundles, which are analogues of holomorphic vector bundles on complex manifolds. For instance, on a complex manifold $M$, generalized holomorphic bundles correspond to co-Higgs bundles, which is a holomorphic vector bundle $E$ on $M$ together with a holomorphic map $\phi : E \to E \otimes T_M$ for which $\phi \wedge \phi = 0$. Some of the general properties of co-Higgs bundles were studied by Hitchin in [14] and moduli spaces of stable co-Higgs bundles were studied in [28, 29, 30, 35], etc. Given the relationship between the generalized complex geometry and the bi-Hermitian geometry, one can study generalized holomorphic bundles in terms of $I_{\pm}$-holomorphic bundles. Recall that any $J$-holomorphic bundle over generalized Kähler manifold $(M, J, J')$ induces an $I_{\pm}$-holomorphic bundle on $(M, g, I_+, I_-)$ (see [16, Proposition 2.11]). We will not list the definitions on generalized complex geometry (see [11, 16] for more details). Therefore, combining Theorem 1.1, we have the following result.

**Corollary 1.3.** Let $(M, J, J')$ be a compact generalized Kähler manifold with non-empty boundary $\partial M$ whose associated bi-Hermitian structure $(g, I_+, I_-)$ is such that $\text{vol}_g = \frac{\omega_{\pm}^n}{n!}$. Moreover, suppose $(E, \overline{\partial}_+, \overline{\partial}_-)$ is a $J$-holomorphic bundle on $M$. Then for any Hermitian metric $\varphi$ on the restriction of $E$ to $\partial M$, there is a unique $\alpha$-Hermitian-Einstein metric $H$ on $E$ such that $H = \varphi$ on $\partial M$.

This paper is organized as follows. In Section 2, we will introduce the $\alpha$-Hermitian-Einstein flow on bi-Hermitian manifolds. And some elementary calculations will be presented. In Section 3, we prove the long-time existence of the $\alpha$-Hermitian-Einstein flow over a compact bi-Hermitian manifold. At last, we deal with convergence of the $\alpha$-Hermitian-Einstein flow over a compact bi-Hermitian manifold with boundary, that is we complete the proof of Theorem 1.1.
Proposition 2.1. Let \( s, t \) be a solution of the flow (2.3), then

\[
\Delta_{\bar{\partial}_+} := -\sqrt{-1} \Lambda_+ \bar{\partial}_+ \partial_+
\]

and

\[
\Delta_{\bar{\partial}, \alpha} := \alpha \Delta_{\bar{\partial}_+} + (1 - \alpha) \Delta_{\bar{\partial}_-}.
\]

We will see later the following proposition plays an important role in our discussion.

**Proposition 2.1.** Let \( H(t) \) be a solution of the flow (2.3), then

\[
(\Delta_{\bar{\partial}, \alpha} - \frac{\partial}{\partial t})|_H \alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_- - \lambda \cdot \text{Id}_E|^2_H \geq 0.
\]

**Proof.** For simplicity, set

\[
\eta = \alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_- - \lambda \cdot \text{Id}_E.
\]

Then from (2.1) and (2.2), we have

\[
\Delta_{\bar{\partial}_\pm} |\eta|^2_H = -\sqrt{-1} \Lambda_\pm \bar{\partial}_\pm \partial_\pm \text{tr}\{\eta H^{-1} \eta^T H\}
\]
\[ \sigma(H, K) = \text{tr}(H^{-1}K) + \text{tr}(K^{-1}H) - 2r, \]
where \( r = \text{rank}(E) \).

If we choose a local frame to diagonalize \( H^{-1}K \) to be \( \text{diag}(\lambda_1, \ldots, \lambda_r) \), then
\[ \sigma(H, K) = \sum_{i=1}^{r} (\lambda_i + \lambda_i^{-1} - 2), \]
from which we can see that \( \sigma \geq 0 \), with equality holds if and only if \( H = K \). Let \( d \) be the Riemannian distance function on the metric space, then
\[ f_1(d) \leq \sigma \leq f_2(d) \]
holds for some monotone functions \( f_1 \) and \( f_2 \). So we can conclude from this inequality that a sequence of metrics \( H_i \) converge to some \( H \) in the usual \( C^0 \)-topology if and only if \( \sup_M \sigma(H_i, H) \to 0 \).

**Proposition 2.3.** Let \( H, K \) be two \( \alpha \)-Hermitian-Einstein metrics, then
\[ \Delta_{\tilde{\alpha}, \alpha} \sigma(H, K) \geq 0. \]

**Proof.** Let \( h = K^{-1}H \), from (2.2) we have
\[ \text{tr}\{\sqrt{-1}\Lambda_{\pm} F^H \pm \Lambda_{\pm} F^K \} = -\Delta_{\tilde{\alpha}} \text{tr} h + \text{tr}\{-\sqrt{-1}\Lambda_{\pm} \bar{\partial}_{\pm} \pm h^{-1} \partial^K_h \}, \]
and
\[ \text{tr}\{\sqrt{-1}h^{-1}(\Lambda_{\pm} F^K \pm \Lambda_{\pm} F^H) \} = -\Delta_{\tilde{\alpha}} \text{tr} h^{-1} + \text{tr}\{-\sqrt{-1}\Lambda_{\pm} \bar{\partial}_{\pm} h^{-1} \partial^K_h h^{-1}\}. \]
On the other hand, by doing calculation locally ([8]), it is easy to check that
\[ \text{tr}( -\sqrt{-1}\Lambda_\pm \bar{\partial}_\pm h h^{-1} \partial^K h ) \geq 0 \]
and
\[ \text{tr}( -\sqrt{-1}\Lambda_\pm \bar{\partial}_\pm h h^{-1} h \partial^K h^{-1} ) \geq 0. \]
Hence we complete the proof. □

Next, instead of considering \( H \) and \( K \) to be \( \alpha \)-Hermitian-Einstein metrics, we assume \( H = H(t) \), \( K = K(t) \) to be two solutions of the \( \alpha \)-Hermitian-Einstein flow (2.3) with the same initial value \( H_0 \). Similar to Proposition 2.3, we can prove the following.

**Proposition 2.4.**
\[ (\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})\sigma(H(t), K(t)) \geq 0. \]

**Proof.** Set \( h(t) = K(t)^{-1}H(t) \). Notice that
\[ \frac{\partial}{\partial t} \text{tr} h = \text{tr}(K^{-1}HK^{-1} \partial^K H - K^{-1} \partial^K KK^{-1} H), \]
\[ \frac{\partial}{\partial t} \text{tr} h^{-1} = \text{tr}(-H^{-1} \partial^K HH^{-1} K + H^{-1} KK^{-1}\partial^K K). \]
These two identities together with Proposition 2.3 show that
\[ (\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})(\text{tr} h + \text{tr} h^{-1}) \geq 0. \]
□

### 3. \( \alpha \)-Hermitian-Einstein flow on compact bi-Hermitian manifold

In this section our primary purpose is to prove the long-time existence of the \( \alpha \)-Hermitian-Einstein flow over a compact bi-Hermitian manifold. When the base manifold \( M \) is closed, we consider the following problem:

\[
\begin{aligned}
\frac{\partial}{\partial t} H & = - (\alpha\sqrt{-1}\Lambda_+ + (1-\alpha)\sqrt{-1}\Lambda_-) + \lambda \cdot \text{Id}_E, \\
H(0) & = H_0.
\end{aligned}
\]

And when \( M \) is a compact manifold with a non-empty smooth boundary \( \partial M \), for any given initial metric \( \varphi \) over \( \partial M \) we instead consider the following boundary value problem:

\[
\begin{aligned}
\frac{\partial}{\partial t} H & = - (\alpha\sqrt{-1}\Lambda_+ + (1-\alpha)\sqrt{-1}\Lambda_-) - \lambda \cdot \text{Id}_E, \\
H(0) & = H_0, \\
H|_{\partial M} & = \varphi.
\end{aligned}
\]

Since (2.3) is non-linear strictly parabolic. So we get the short-time existence from the standed parabolic PDE theory [12].

**Theorem 3.1.** For sufficiently small \( \varepsilon > 0 \), the problem (3.1) and (3.2) have a smooth solution defined for \( 0 \leq t < \varepsilon \).

Next, following a standard argument, we can show the long-time existence of (3.1) and (3.2).
Lemma 3.2. Suppose that a smooth solution $H_t$ to (3.1) or (3.2) is defined for $0 \leq t < T$. Then $H_t$ converges in $C^0$ topology to some continuous non-degenerate metric $H_T$ as $t \to T$.

Proof. In order to prove the convergence, it suffices to show that, given any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\sup_M \sigma(H_t, H_{t'}) < \varepsilon, \quad \text{for all } t, t' > T - \delta.$$ 

And this can be easily seen from the continuity at $t = 0$ combining with Proposition 2.4 and the maximum principle.

So, it remains to show $H_T$ is non-degenerate. By Proposition 2.1 we know that

$$\sup_{M \times [0, T)} |\alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_- - \lambda \cdot \Id_E|_H^2 < C,$$

where $C = C(H_0)$ is a uniform constant. By a direct calculation we have

$$\left| \frac{\partial}{\partial t} (\ln tr h) \right| = \left| \text{tr} \left\{ h \left( \alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_- - \lambda \cdot \Id_E \right) \right\} \frac{1}{\text{tr} h} \right|_H$$

$$\leq \left| \alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_- - \lambda \cdot \Id_E \right|_H.$$ 

And similarly

$$\left| \frac{\partial}{\partial t} (\ln tr h^{-1}) \right| \leq \left| \alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_- - \lambda \cdot \Id_E \right|_H.$$ 

Then we can conclude that $\sigma(H, H_0)$ are uniformly bounded on $M \times [0, T)$, which implies that $H_T$ is non-degenerate. \hfill \Box

For further consideration, we prove the following lemma in the same way as [8, Lemma 19] and [31, Lemma 6.4] (also see [37, Lemma 3.3]).

Lemma 3.3. Suppose $(M, g, I_+, I_-)$ is a closed bi-Hermitian manifold without boundary (compact with non-empty boundary). Let $H(t)$, for $0 \leq t < T$, be a one-parameter family of Hermitian metrics on $(E, \bar{\partial}+, \bar{\partial}-)$ over $M$ (satisfying the Dirichlet boundary condition), such that

(i) $H(t)$ converges in $C^0$ topology to some continuous metric $H_T$ as $t \to T$, 
(ii) $\sup_M |\alpha \sqrt{-1} \Lambda_+ F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda_- F^H_-|_{H_0}$ is uniformly bounded for $t < T$.

Then $H(t)$ is bounded in $C^1$, and also bounded in $L^p_\Sigma$ (for any $1 < p < \infty$) uniformly in $t$.

Proof. Let us first follow Donaldson’s arguments [8, Lemma 19]. Let $h(t) = H_0^{-1} H(t)$. We claim that $h(t)$ are bounded uniformly in $C^1$. If this not true, then for some subsequence $t_j$ there are points $x_j \in M$ with $\sup |dh_j|_{x_j} = l_j$ achieved at $x_j$, and $l_j \to \infty$, here we denoted by $h_j = h(t_j)$.

(a) When $M$ is a closed manifold. We can suppose that the $x_j$ converges to a point $x$ in $M$ after taking a subsequence. Once choosing local coordinates $\{z_\alpha\}_{\alpha=1}^n$ around $x_j$ and translating the coordinates slightly, we can suppose that

$$\sup |dh_j|_{x_j} = l_j$$

is attained at $z = 0$. Rescale $\{z_\alpha\}$ to new coordinates $\{w_\alpha\}$ by $w_\alpha = l_i z_\alpha$; that is, via the maps $\{|w_\alpha| < 1\} \to \{|w_\alpha| < l_j^{-1}\}$, pull back the matrices $h_j$ to matrices $\tilde{h}_j$
defined for \(|w_\alpha| < 1\). With respect to the rescaled coordinates,

\[
\sup_{|w_\alpha| < 1} |d\tilde{h}_j| = 1
\]

is attained at the origin point. For convenience, we set \(\partial^0_\pm := \partial^H_\pm, \tilde{F}^i_\pm := \tilde{F}^H(t_j)\).

Under the assumption of the lemma, we have

\[
|\alpha\sqrt{-1}\Lambda_+ \tilde{F}^i_+ + (1 - \alpha)\sqrt{-1}\Lambda_- \tilde{F}^i_- - \alpha\sqrt{-1}\Lambda_+ \tilde{F}^0_+ - (1 - \alpha)\sqrt{-1}\Lambda_- \tilde{F}^0_-| \\
= |\alpha\tilde{h}^{-1}_j (\Lambda_+ \partial_+ \partial^0_+ \tilde{h}_j - \Lambda_- \partial_- \tilde{h}_j \tilde{h}^{-1}_j \partial^0_+ \tilde{h}_j) \\
+ (1 - \alpha)\tilde{h}^{-1}_j (\Lambda_- \partial_- \partial^0_- \tilde{h}_j - \Lambda_- \partial_- \tilde{h}_j \tilde{h}^{-1}_j \partial^0_- \tilde{h}_j)|
\]

is bounded in \(\{|w_\alpha| < 1\}\). Since \(\tilde{h}_j\) and \(d\tilde{h}_j\) are bounded, \(|\Lambda_\pm \partial_\pm \partial^0_\pm \tilde{h}_j|\) are bounded independent of \(j\), then \(|\Delta_{\partial_\alpha} \tilde{h}_j|\) is also bounded independent of \(j\). By the properties of the elliptic operator \(\Delta_{\partial_\alpha}\) on \(L^p\) spaces, \(\tilde{h}_j\) are uniformly bounded in \(L^p_\alpha\) on a small ball. After taking \(p > 2n\), \(L^p_\alpha \rightarrow C^1\) is compact, then we deduce that some subsequence of the \(\tilde{h}_j\) converge strongly in \(C^1\) to \(\tilde{h}_\infty\). But on the other hand the variation of \(\tilde{h}_\infty\) is zero, since the original metrics approached a \(C^0\) limit, which contradicts the fact

\[
|d\tilde{h}_\infty|_{z=0} = \lim_{j \to \infty} |d\tilde{h}_j|_{z=0} = 1.
\]

(b) When \(M\) is a compact manifold with non-empty boundary \(\partial M\). We will adapt Simpson’s arguments \([31, \text{Lemma 6.4}]\) to our settings. Let \(d_j\) denote the distance from \(x_j\) to the boundary \(\partial M\), then there are two cases.

(b1) If \(\limsup d_j l_j > 0\), then we can choose balls of radius \(\leq d_j\) around \(x_j\) and rescale by a factor of \(l_j^{-1/2}\) to a ball of radius 1 (where \(\epsilon < \lim sup d_j l_j\)), and pull back the matrices \(h_j\) to matrices \(\tilde{h}_j\) defined on \(\{|w_\alpha| < 1\}\). With respect to the rescaled coordinates,

\[
\sup |d\tilde{h}_j| = \epsilon
\]

is attained at the origin. By condition of the lemma, and discussing like that in (a), we will deduce contradiction.

(b2) On the other hand, if \(\limsup d_j l_j = 0\), we can assume that \(x_j\) approach a point \(y\) on the boundary. Then let \(\tilde{x}_j \in \partial M\) so that \(\text{dist}(\tilde{x}_j, x_j) = d_j\), also \(\tilde{x}_j\) approach \(y\). Choose half-balls of radius \(l_j^{-1/2}\) around \(\tilde{x}_j\) and rescale by \(l_j\) to the unit half-balls. In the rescaled picture, \(x_j\) approach \(z = 0\). In the rescaled coordinates, \(|\Lambda_\pm \partial_\alpha \partial^0_\pm \tilde{h}_j|\) are still bounded, \(\tilde{h}_j\) are uniformly bounded, and \(\sup |d\tilde{h}_j| = 1\).

Since \(\tilde{h}_j\) satisfy boundary condition along the face of the half-ball, then using elliptic estimates with boundary, and discussing like that in (a), we can also deduce contradiction.

From the above discussion, \(h(t)\) are uniformly bounded in \(C^1\). Using (3.3) together with the bounds on \(h(t), |\alpha\sqrt{-1}\Lambda_+ \tilde{F}^i_+ + (1 - \alpha)\sqrt{-1}\Lambda_- \tilde{F}^i_-|, \) and \(dh\) imply that \(\Lambda_\pm \partial_\alpha \partial^0_\pm h\) are uniformly bounded. By the elliptic estimates with boundary conditions, \(h(t)\) are uniformly bounded in \(L^p_\alpha\) for any \(1 < p < \infty\).

\[\square\]

**Theorem 3.4.** (3.1) and (3.2) have a unique solution \(H(t)\) which exists for \(0 \leq t < \infty\).
Proof. By Theorem 3.1, we suppose that there is a solution \( H(t) \) existing for \( 0 \leq t < T \). By Lemma 3.2, \( H(t) \) converges in \( C^0 \) topology to a continuous non-degenerate metric \( H_T \). This together with the fact that \( \sup_M |\alpha \sqrt{-1} \Lambda + F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda - F^H_-|_{H_0} \) is bounded uniformly in \( t \) implies that \( H(t) \) are bounded in \( C^1 \), and also bounded in \( L^p \) (for any \( 1 < p < \infty \)) uniformly in \( t \). Since (3.1) and (3.2) are quadratic in the first derivative of \( H \), we can apply Hamilton’s method \[12\] to deduce that \( H(t) \) are bounded in \( C^\infty \), and the solution can be extended past \( T \).

Hence we have showed the long-time existence of problem (3.1) and (3.2). As for the uniqueness, one can easily achieve it from Proposition 2.4 and the maximum principle. \( \square \)

4. Proof of Theorem 1.1

Since we have proved the long-time existence of (3.2), it remains for us to show that the solution \( H(t) \) converges to a metric \( H_\infty \) as the time \( t \) approaches to the infinity, and that the limit \( H_\infty \) is \( \alpha \)-Hermitian-Einstein.

Suppose \( H(t) \) is a solution to (3.2) for \( 0 \leq t < \infty \). As in the previous section we still set
\[
\eta = \alpha \sqrt{-1} \Lambda + F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda - F^H_- - \lambda \cdot \text{Id}_E.
\]

From Proposition 2.1 and the fact that \( |\theta|^2_{H} \leq |\theta|^2_{H} \) holds for any section \( \theta \) of \( \text{End}(E) \), we have
\[
(\Delta_{\bar{\partial}, \alpha} - \frac{\partial}{\partial t}) |\eta|_{H} \geq 0.
\]

Next, according to the Proposition 1.8 of Chapter 5 in [33], the following Dirichlet problem is solvable:
\[
\begin{align*}
\Delta_{\bar{\partial}, \alpha} v &= -\left|\alpha \sqrt{-1} \Lambda + F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda - F^H_- - \lambda \cdot \text{Id}_E\right|_{H_0}, \\
v|_{\partial M} &= 0.
\end{align*}
\]

Then set \( w(x, t) = \int_{0}^{t} |\eta|_{H}(x, s)ds - v(x) \), where \( v(x) \) is a solution to the problem above. From (4.1), (4.2) and the boundary condition satisfied by \( H \) we can see that for \( t > 0 \), \( |\eta|_{H} \) vanishes over the boundary \( \partial M \). Then one can easily check that \( w(x, t) \) satisfies
\[
\begin{align*}
(\Delta_{\bar{\partial}, \alpha} - \frac{\partial}{\partial t}) w(x, t) &\geq 0, \\
w(x, 0) &= -v(x), \\
w(x, t)|_{\partial M} &= 0.
\end{align*}
\]

Therefore the maximum principle implies that
\[
\int_{0}^{t} |\eta|_{H}(x, s)ds \leq \sup_{y \in M} v(y),
\]
for any \( y \in M \) and \( 0 \leq t < \infty \).

Let \( 0 \leq t_1 \leq t < \infty \), \( \tilde{h} = H^{-1}(x, t_1)H(x, t) \). Obviously \( \tilde{h} \) satisfies
\[
\tilde{h}^{-1} \frac{\partial}{\partial t} \tilde{h} = -\left(\alpha \sqrt{-1} \Lambda + F^H_+ + (1 - \alpha) \sqrt{-1} \Lambda - F^H_- - \lambda \cdot \text{Id}_E\right) = -\eta.
\]

Then we have
\[
\frac{\partial}{\partial t} \ln \text{tr} \tilde{h} \leq 2 |\eta|_{H}.
\]
Integrating it over \([t_1, t]\) gives
\[
\text{tr} \tilde{h} = \text{tr} (H^{-1}(x, t_1)H(x, t)) \leq r \exp \left(2 \int_{t_1}^t |\eta|_H \, ds \right).
\]

Treating \(\tilde{h}^{-1}\) in the same way one can get a similar estimate for it. Combining them together we can conclude that
\[
\sigma (H(x, t), H(x, t_1)) \leq 2r \left( \exp(2 \int_{t_1}^t |\eta|_H \, ds) - 1 \right).
\]

From (4.4) and (4.5), we have that \(H(t)\) converges in the \(C^0\) topology to some continuous metric \(H_\infty\) as \(t \to +\infty\). Hence using Lemma 3.3 again we know that \(H(t)\) has uniform \(C^1\) and \(L^2\) bounds. This together with the fact that \(|H^{-1} \frac{\partial}{\partial t} H|\) is uniformly bounded and the standard elliptic regularity arguments shows that, by passing to a subsequence if necessary, \(H(t) \to H_\infty\) in \(C^\infty\) topology. And from (4.4) we have
\[
\alpha \sqrt{-1} \Lambda \cdot F_+^{\infty} + (1 - \alpha) \sqrt{-1} \Lambda \cdot F_-^{\infty} - \lambda \cdot \text{Id}_E = 0,
\]
i.e. \(H_\infty\) is the desired \(\alpha\)-Hermitian-Einstein metric satisfying the Dirichlet boundary condition. The uniqueness of the solution comes from Proposition 2.3 and the maximum principle. Hence we complete the proof of Theorem 1.1.

REFERENCES


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