On the arrow of spacetime

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Consistent with special relativity and statistical physics, here we construct a partition function of space-time events. The union of these two theories resolves longstanding problems in regard to time. It augments the standard description of time given by the (non-relativistic) arrow of time to one able to show the emergence of three macroscopic regimes of time: the past, the present, and the future, represented by space-like entropy, light-like entropy, and time-like entropy, respectively, and in a manner consistent with our experience of said regimes. First, using Fermi-Dirac statistics, we find that the system essentially describes a "waterfall" of space-time events. This "waterfall" recedes in space-time at the speed of light towards the direction of the future as it "floods" local space with events that it depletes from the past. In this union, an observer $O$ will perceive two horizons that can be interpreted as hiding events behind it. The first is an event horizon, and its entropy hides events in the regions that $O$ cannot see. The second is a time horizon, and its entropy "shields" events from $O$'s causal influence. As only past events are "shielded", and not future events, an asymmetry in time is thus created. Finally, future events are hidden by an entropy prohibiting $O$ from knowing the future before the present catches on.

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1 Introduction

1.1 Time

The connection between time in statistical physics and time in special relativity are both well established (Popper [1956], Layzer [1975], Davies [1977], Zeh and Page [1990], Callender [2001], Price [1997], Landsberg [1972], Halliwell et al. [1996], Carroll [2010], Zeh and Page [1990]). In the one hand, we have the statistical emergence of a macroscopic arrow of time, and in the other, we have a causal relationship between space-time events limited by the speed of light. As most other theories consider time to be little more than some reversible variable, the claims made in regard to time by these two theories pack quite a lot of heat (figuratively). Thus, to further investigate the nature of time, a promising avenue might be to ask; what about time in "statistical physics $\cup$ special relativity" — is it possible to combine causality (in the sense given by special relativity) with the arrow of time? As it turns out, doing so we are rewarded with quite a lot of insight. First, let’s clarify the motivation.

1.2 Problem 1: The direction of time

We recall the well-known thought experiment in which Maxwell’s demon (Maxwell [1891], Knott [1911], Thomson [1879], Leff and Rex [2002]) consumes good information to return a macroscopic system to some initial low-entropy state. An observer might naively conclude that the (macroscopic) arrow of time (Weinert [2004]) is reversed, but inspection of the global system reveals that Maxwell’s demon produced entropy in amounts at least equal to the information consumed (Szilard [1929], Landauer [1961], Bennett [2003]).

Taking the universe to be the global system, and sub-systems to be local, equivalent thought experiments can be produced. Borrowing terms from special relativity, we might say that the arrow of time holds globally, but not locally. Gotcha! A global property in a non-relativistic theory is bound to violate causality.

Or as George F R Ellis (Ellis [2013]) states:

"The arrow of time locality issue: If there is a purely local process for determining the arrow of time, why does it give the same result everywhere? [...] Local determination has to arbitrarily choose one
of the two directions of time as the positive direction indicating the future; but as this decision is made locally, there is no reason whatever why it should be consistent globally. If it emerges locally, opposite arrows may be expected to occur in different places.

Resolving this issue is, of course, an interesting result of the model. In the model, the arrow of space-time cannot be reversed by Maxwell’s demon, even locally, because the space-like entropy of the sub-system, associated with the system’s past, grows throughout the cycle proportionally to the speed of light. The causal influence of the system (which propagates at the speed of light in all directions), is, in this model, the macroscopic system that would need to be reversed by Maxwell’s demon to reverse the arrow of time. Thus, Maxwell’s demon cannot succeed unless it violates the speed of light—checkmate!

1.3 Problem 2: Macroscopic time

If we were to only rely upon time-symmetric differential equations (e.g., classical mechanics, unitary evolution, etc.), there would be no meaningful difference between the past, the present, and the future. Basically, time would be a point on a line, and we could solve our equations as far into the future or into the past as we wanted. This is quite far from our day-to-day experience of time.

Consistent with our macroscopic experience of time, the arrow of time improves the connection between "time in the equations" and "time in real life" by giving it a direction favored by entropy. This is better, but, as we will argue, not yet complete.

We will now scope our work by formulating some contemplation questions to elucidate what is missing from the arrow of time, along with some leading questions to guide the discussion towards the solution presented here. We denote the rate of increase in entropy over time as \( \frac{dS}{dt} \), and an observer by \( O \). Let’s start with two questions specifically related to the arrow of time:

1. A system can increase its entropy quickly \( \left( \frac{dS}{dt} \gg 0 \right) \) or slowly \( \left( \frac{dS}{dt} \rightarrow 0^+ \right) \), and in each case it will see time pass at the same rate (assuming the same frame of reference). Why is the rate of passage of time not proportional to \( \frac{dS}{dt} \) if its direction is correlated to it?

2. Special relativity sets the magnitude for the time evolution of \( O \) in its own frame of reference to be proportional to the speed of light. Leading question: why is the direction of time set by statistical physics, but its magnitude is set by special relativity; two seemingly unrelated theories?
Furthermore, there is a noticeable difference between the past, the present, and the future that is not captured by the arrow of time. An image that I like to use is that of an \( O \) who regrets its past, worries about its future, and acts (or fails to act) in the present. Consistent with causality, \( O \) ought not to experience the other permutations, such as regretting the future or worrying about the past.

Or, again, as George F R Ellis (Ellis [2013]) states:

"As The nature of existence is different in the past and in the future - Becoming has meaning. Different ontologies apply in the past and future, as well as different epistemologies"

With this image in mind, a further series of questions arises:

1. Suppose \( O \) goes bird watching. Say the birds are observed for 20 years by \( O \). Then \( O \) should know a lot more about those birds today than \( O \) did 20 years ago. \( O \)’s conclusion that time has passed is evidenced by the existence of a "log of observations" maintained by \( O \), and not so much on the fact that the birds of today are ever so slightly closer to final cosmological thermodynamic equilibrium than the birds of 20 years ago. Let’s emphasize the dichotomy; the arrow of time points towards higher entropy (more hidden information) but \( O \) concluded that time moves forward by building up a log of events (gaining information). Is \( O \) making a mistake by using a log of events to conclude as to the passage of time, instead of estimating the system’s overall entropy as per the arrow of time? As George F R Ellis puts it, "we are aware of the flow of time because of the existence of records of the past" (Ellis [2013]). Leading question: Is \( O \)’s construction of a log of events contributing to increasing the entropy of the system? What if \( O \)’s log is complete (it lists all space-time events) — in this case, is \( O \)’s behavior of producing information by increasing entropy the opposite of Maxwell’s demon? Perhaps it is "Maxwell’s angel”? (Touchette and Lloyd [2000], Parrondo et al. [2015])?

2. Generally, if the universe started from a low entropy big bang and is evolving into a high entropy state, and that entropy is associated with a loss of information, what sort of compensation is there to offset this loss? Leading question: Can the information gained by observing a system over time be accumulated, and is it enough to offset the information lost by the action of the arrow of time? Can a statistical system aware of events reconcile both notions? From Landauer’s aphorism, since “information is physical” (Landauer et al. [1991]) where and how would his information be stored(Goold et al. [2016], Streltsov et al. [2017], Adesso et al. [2016])?
3. Why is the causal connection between past, present and future asymmetric (Halliwell et al. [1996])? For example, \( O \) opens the fridge and notices that there is milk in it. \( O \) wonders why the milk is there. It would make no sense to appeal to the future to justify the presence of milk in the present. Thus saying, “the milk is in the fridge because \( O \) will drink it tomorrow and therefore it must be here today for \( O \) to be able to drink it tomorrow — that’s why it’s there!” makes no sense because \( O \) could simply elect not to drink the milk and the milk will still be in the fridge today. Its similar to how some people feel absolutely certain that they will win the lottery at the next draw, and thus will buy a ticket today in an attempt to set up the present to be as consistent as possible with the guaranteed future winnings, only to witness the attempt fail. However, the reverse explanation easily makes sense; the milk is in the fridge today because \( O \) bought it at the supermarket yesterday and put it in the fridge this morning. Why does the “algorithmic reconstruction” of the past based on present evidence work, but the same reconstruction fails when applied to the future? Using purely time-symmetric differential equations, we would get a different story: the present would fix both the past and the future! We could even appeal to the future to justify the present. The “arrow of causality” that we experience fails to emerge from exactly solvable time-symmetric differential equations.

4. From \( O \)’s present, the future appears to have multiple outcomes that could occur (e.g., \( O \) could elect to drink or not to drink the milk — we won’t know until tomorrow) (Eddington [2012]). Leading question: Is there an entropy that we could have missed, but is nonetheless associated with \( O \)’s possible futures that somehow gets reduced as \( O \) travel forward in time, as the present unravels? Does this entropy prohibit \( O \)’s knowledge about the future?

Combining special relativity with statistical physics will allow us to account for the past, the present, and the future as three distinct regimes of time in a manner consistent with experience and with the points raised in these questions. Let us now describe in greater details how this will be done.

1.4 Outline

We consider two approaches that we could use to combine statistical physics with special relativity.

One approach is to make the standard quantities of statistical physics (e.g., entropy, temperature) Lorentz-invariant. Using this methodology, we are rewarded with a frame-independent partition
function. The result of this approach yields non-relativistic statistical physics in the limit $c \to \infty$. This is the approach taken by Giorgio Kaniadakis (Kaniadakis [2002]).

Another approach is to seek a statistical ensemble whose macroscopic description relates to special relativity. As special relativity is concerned with the relationship between space-time events, this approach elects to make special relativity emergent (via an equation of state) from a statistical ensemble of space-time events. In this case, special relativity is an emergent behavior caused by the random statistics of space-time events.

The latter approach is the one that will be taken in this paper. Not only does it paint a seductive picture of time and space, but it is also attractive because of its ability to explain the nature of macroscopic time.

Using Fermi-Dirac statistics over an ensemble of space-time events, we show (section 3.1) that the space-time counterpart to the arrow of time is a “waterfall of events”. Like the arrow of time, it too has a direction; it recedes in space-time at the speed of light towards the direction of the future (section 3.2). Furthermore, as it recedes, the waterfall of events floods the present with immediate events and depletes the past of events. As we will see, this behavior connects to the three regimes of time (section 3.3 and 4.1).

2 A statistical ensemble of events

Our goal for this section will be to recover the "features" of special relativity strictly using the facilities of statistical physics. In this case, we would say that special relativity is an emergent property of the constructed statistical ensemble. How will we do that? First, we have to interpret the speed of light as a tool to hide information. Specifically, the speed of light hides information regarding events whose intervals to the observer are space-like. Interpreted as such, we can then use the entropy in statistical physics to achieve the same purpose as the speed of light (hide information), provided that we "place" the entropy appropriately in the system.

For instance, we can imagine an observer $O$ whose causal horizon is defined by the usual light cone of special relativity. We can then bring the description back to statistical physics simply by asking; how many configurations of space-time events outside the light cone are compatible with the inside? Answer: since no information can be obtained about the outside, we must conclude that all configurations of outside events are compatible with the inside. Consequently, as far as $O$ is concerned, the events outside $O$'s light cone must be of maximum entropy or information about them will be available to $O$. 
Attributing an entropy to events separated by an horizon in order to connect to thermodynamics (Bekenstein [1973]) has been done since at least 1973 by J.D. Bekenstein. Furthermore, from G. W. Gibbons and S. W. Hawking’s 1977 article (Gibbons and Hawking [1977]), I quote:

"An observer in these models will have an event horizon whose area can be interpreted as the entropy or lack of information of the observer about the regions which he cannot see."

The part missing from the picture of time, we suggest, is to apply the same line of reasoning to configurations of future events. We ask, how many configurations of future events are compatible with the present state? First, we pose this assumption: since the present is caused by the past (not the future), there exists strictly more than one configuration of future states compatible with the current present. The use of the quantifier strictly more than one is important here as if it were not the case (i.e., there is only one configuration of future events), then the present would be “caused/determined” by the future as much as by the past. Causality would be time-symmetric. This assumption is the foundation of the past/future asymmetry we recover in this union. For instance, an observer trying to pre-solve time-symmetric exact equations will, due to this entropy, fail to predict which of the possible futures will actually occur. However, once the future has become present, solving those same equations backward will go as expected. We, of course, assume that an observer has no prior knowledge of his future (no foresight). Thus, configurations of compatible future events must be of maximum entropy or information about the future will be available to \( O \) (foresight).

Using this strategy, we can obtain a system of statistical physics over space-time events that follows the rules of special relativity.

Other emergent features we would like to have (briefly listed and to be expanded in section 3) are: a) recovering the fundamental equation of special relativity linking time to space \( dx = c dt \) from the equation of state; b) having the entropy at the horizon corresponding to the Bekenstein-Hawking entropy \( S = k_B A / (4 L_p^2) \); c) have the speed of light defined as an emergent property of the system, which is constant and that cannot be exceeded; d) connect time-like separated events with a "time-like entropy" and space-like separated events with a "space-like entropy"; e) have an emergent arrow of space-time that generalizes the arrow of time to special relativity. This list of requirements might sound like the statistical system would be complicated to describe, but all it takes is maximizing the entropy on average \( t \) (system age) and \( x \) (system size), and everything we need will emerge out of it. Let’s get started.
2.1 Background

We suppose a 3+1 space-time \( M_4 \) in spherical coordinates \( \{r, \theta, \phi, t\} \). Under isotropic assumptions \( \{d\theta = 0, d\phi = 0\} \) the space-time is simplified to a 1+1 space-time \( M \) with coordinates \( \{r, t\} \), and it enforces the constraint that \( r \in \mathbb{R}_{\geq 0} \). Additionally, we pose that \( t = 0 \) is the origin of the system, and thus \( t \in \mathbb{R}_{\geq 0} \). Finally, we denote an observer by \( O \).

2.2 Events

Let \( Q \) be the set of events in \( M \). We define the functions \( r \) and \( t \) as mapping each event \( q \in Q \) to its space-time position in \( M \) as:

\[
\begin{align*}
  r : Q &\rightarrow \mathbb{R}_{\geq 0} \text{ [meters]} \\
  t : Q &\rightarrow \mathbb{R}_{\geq 0} \text{ [seconds]}
\end{align*}
\]

2.3 Macroscopic system of events

We consider a macroscopic system defined for a set of events \( Q \) and two macroscopic quantities (the priors): an average event-time \( \bar{t} \in \mathbb{R}_{\geq 0} \), and an average event-distance \( \bar{r} \in \mathbb{R}_{\geq 0} \).

2.4 Gibbs ensemble of events

Under the principle of maximum entropy, we seek the probability distribution \( \rho : Q \rightarrow \{p \in \mathbb{R} | 0 \leq p \leq 1\} \) and \( \sum_{q \in Q} \rho(q) = 1 \) which maximizes the entropy \( S \):

\[
S = -k_B \sum_{q \in Q} \rho(q) \ln \rho(q)
\]

and subject to the priors \( \bar{t} \) and \( \bar{r} \)

\[
\begin{align*}
  \bar{t} &= \sum_{q \in Q} \rho(q)t(q) \\
  \bar{r} &= \sum_{q \in Q} \rho(q)r(q)
\end{align*}
\]

We maximize the entropy using the well-known method of the Lagrange multipliers.
The action functional is given by:

\[ \mathcal{L} = \left( -k_B \sum_{q \in Q} \rho(q) \ln \rho(q) \right) + \lambda_0 \left( \sum_{q \in Q} \rho(q) - 1 \right) \\
+ \lambda_1 \left( \sum_{q \in Q} \rho(q)t(q) - \bar{t} \right) + \lambda_2 \left( \sum_{q \in Q} \rho(q)r(q) - \bar{r} \right) \]  

(6)

Maximizing \( \mathcal{L} \) with respect to \( \rho(q) \) is done by taking its derivative and posing it equal to zero:

\[ \frac{\partial \mathcal{L}}{\partial \rho(q)} = -k_B \ln \rho(q) - k_B + \lambda_0 + \lambda_1 t(q) + \lambda_2 r(q) = 0 \]  

(7)

Solving for \( \rho(q) \) we obtain:

\[ \rho(q) = \exp \left( \frac{k_B + \lambda_1 + \lambda_2 t(q) + \lambda_3 r(q)}{k_B} \right) \]  

(8)

From the constraint \( 1 = \sum_{q \in Q} \rho(q) \), we can find the value for \( \lambda_0 \):

\[ 1 = \sum_{q \in Q} \rho(q) \]  

(9)

\[ = \sum_{q \in Q} \exp \left( \frac{-k_B + \lambda_0 + \lambda_1 t(q) + \lambda_2 r(q)}{k_B} \right) \]  

(10)

\[ = \exp \left( \frac{-k_B + \lambda_0}{k_B} \right) \sum_{q \in Q} \exp \left( \lambda_1 t(q) + \lambda_2 r(q) \right) \]  

(11)

We define the partition function \( Z \) to be

\[ Z := \sum_{q \in Q} \exp \left( \lambda_1 t(q) + \lambda_2 r(q) \right) \]  

(12)

Then, we rewrite \( \rho(q) \) using \( Z \), we pose \( \lambda_1 := 1/t_0 \) and \( \lambda_2 := -1/r_0 \) and we obtain the probability distribution:

\[ \rho(q) = \frac{1}{Z} \exp \left( \frac{1}{t_0} t(q) - \frac{1}{r_0} r(q) \right) \]  

(13)

where \( 1/t_0 \) with units \([1/\text{(seconds)}]\) and \(-1/r_0 \) with units \([1/\text{(meters)}]\) are the Lagrange multipliers (a justification for the choice of signs will be provided after the results in section 4.3). Finally, we obtain the equation of state:

\[ dS = -\frac{k_B}{t_0} dt + \frac{k_B}{r_0} dr \]  

(14)
which represent the macroscopic evolution of the system, and where \( k_B \) is the Boltzmann’s constant. The reason why the author has elected to produce the explicit derivation of the Gibbs ensemble for this system is to show clearly that a Gibbs ensemble of statistical physics (such as the one here) can legitimately be constructed without the introduction of an emergent temperature (as a Lagrange multiplier) associated with thermodynamic equilibrium. Thus, the present system holds outside of thermodynamic equilibrium, although another type of equilibrium is required on the \( 1/t_0 \) and \(-1/r_0\) Lagrange multipliers (perhaps the name "tempo-dynamic equilibrium" is fitting?) In this case, the Lagrange multipliers \( 1/t_0 \) and \(-1/r_0\) would be the "tempo-ture" of the system.

**Definition:** We will define "light-like" entropy, "time-like" entropy, and "space-like" entropy. Each is obtained by solving \( dS \) in (14) but under different conditions. The first refers to \( dS \) at tempo-dynamic equilibrium \( (1/t_0 dt = 1/r_0 dr) \). The second refers to the case where \( 1/t_0 dt > 1/r_0 dr \), and the third to \( 1/t_0 dt < 1/r_0 dr \).

**Remark:** Since the universe is not at uniform temperature, cosmological thermodynamics has been focused on the study of event horizons, which admits temperatures (Unruh [1976], Hawking [1974]). Resisting the temptation to include an average \( T \), and thus relaxing the requirement that the system be at thermodynamic equilibrium with a uniform temperature \( t \) was a key insight which opened the door to apply statistical physics away from the surface of horizons, and within the volume of the enclosing surface. This is possible at "tempo-dynamic" equilibrium. As we will see, instead of admitting a uniform temperature (as in the thermodynamic equilibrium case), a system at tempo-dynamic equilibrium admits a uniform maximum speed.

3 Results

3.1 Fermi-Dirac statistics of events

We consider that an event can occur at most once (whatever happens to Schrödinger’s cat, for sure, it doesn’t die twice), and thus we will use Fermi-Dirac statistics to study the occupancy distribution of events. In the case of (14), its Fermi-Dirac distribution under the assumption that \( \mu = 0 \) is:

\[
n(q,t_0,r_0) = \frac{1}{\exp \left( \frac{1}{t_0} t(q) - \frac{1}{r_0} r(q) \right) + 1}
\]  

(15)
To better understand what is going on with this equation (Fermi-Dirac statistic over space and time quantities), it helps to first illustrate the Fermi-Dirac statistics of both $r(q)$ and $t(q)$ in isolation. Therefore, consider these Fermi-Dirac distributions applicable to $r(q)$ and $t(q)$, respectively. In each case, one of the two quantities has been made constant for the purposes of simplification. The two distributions (16) and (17) are illustrated in Figure 1 and 2, respectively. The equations are:

$$n(q, x_0) = \frac{1}{\exp \left( \frac{\tau_r - \frac{1}{\tau_0} r(q)}{1} \right) + 1}$$  \hspace{1cm} (16)$$

$$n(q, t_0) = \frac{1}{\exp \left( \frac{\frac{1}{\tau_0} t(q) - \tau_t}{1} \right) + 1}$$  \hspace{1cm} (17)$$

Now we are ready to investigate the Fermi-Dirac distribution given in (15) as illustrated in Figure 3.

Description: For the future, the occupancy rate of events is saturated (future events are upcoming). For the past, the occupancy rate of events is depleted (past events are gone). An observer $O$ evolving from $P_1$ to $P_2$ will, in $O$’s present, experience a transfer in the saturation of the occupancy of future events to a saturation in the occupancy of events in space. This is the point at which we introduce the analogy of the waterfall of events flooding space. Along with $O$,
On the arrow of spacetime, this “waterfall of events” recedes in space-time at the speed of light towards the direction of the future as it “floods” local space with events. \( O \) is prohibited by entropy from knowing future events until the waterfall recedes appropriately. Past events are depleted from the system following the passage of the waterfall. Since \( O \) sees no occupied past events, \( O \)’s future in space-time lies in the forward time direction, and the waterfall recedes towards the future. Let’s prove this direction in the next section, and then return to this discussion in section 4.1.

3.2 Arrow of time

Our goal here is to prove that the waterfall recedes in the direction of the future.

Let us first investigate the production of entropy over time in (14). To do so, we divide each side of (14) by \( d\bar{t} \) then multiply it by \( r_0/k_B \). We obtain:

\[
\frac{r_0}{k_B} \frac{dS}{d\bar{t}} = \frac{r_0}{t_0} - \frac{d\bar{r}}{d\bar{t}}
\]  

(18)

First, we note that as both \(-1/r_0\) and \(1/t_0\) are Lagrange multipliers, then both \(r_0\) and \(t_0\) are uniform in the system. The ratio \(r_0/t_0\) is a speed \([\text{meters/seconds}]\) and is also uniform.

Second, we note that (18) represents an inflection point in the production of entropy in the system at \( dS/d\bar{t} = 0 \). Specifically, when \( dS/d\bar{t} < 0 \), the entropy of the system decreases as a whole, which is prohibited by the second law of thermodynamics. This occurs if the macroscopic system \((\bar{r} \text{ and } \bar{t})\) shrinks \( (d\bar{r}/d\bar{t} < 0) \) or grows slower than the ratio \(r_0/t_0\).

Discussion: We now conclude, based on the two points mentioned, that the arrow of time points towards the future of the macroscopic system whose growth in space-time is prohibited by the second law of thermodynamics from being less than the ratio of the tempo-ture. Now that we know the minimum growth rate, we might wonder, what, if anything, prevents the system from exceeding the ratio? Answer: the observer does not see events beyond the horizon because the occupancy probability, given by Fermi-Dirac statistics, sharply drops to zero at the horizon. This limits the growth rate of the system as perceived by \( O \) to the ratio of its tempo-ture.

3.3 Entropic surface

Let us calculate the entropy at the horizon for the system then we will be in a good position to give a physical interpretation to the tempo-ture.
We can show that the entropy at the horizon is no greater than the Bekenstein-Hawking entropy (Bekenstein [1973], Hawking [1974], Susskind [1995]). To derive it, we must be consistent with the conditions permitting the derivation of the Bekenstein-Hawking entropy in the first place: event horizons admit a temperature (Unruh [1976], Hawking [1974]).

First, we pose \( d\bar{t} = 0 \). Then, the first step of the proof will involve taking the system at tempo-dynamic equilibrium (14) and making it into a system that is also at thermodynamic equilibrium. To do so, we must insert a temperature into the system while keeping in mind that \( 1/r_0 \) is a Lagrange multiplier and, thus, is uniform at equilibrium. Preserving the units while injecting a temperature, we pose the relation \( k_B/r_0 = F/T \) (where \( F \) is a force in \([\text{Joules/meters}]\) and \( T \) is a temperature in \([\text{Kelvin}]\)), and we get:

\[
dS = \frac{F}{T} d\bar{r}
\]  

(19)

For the system to be at thermodynamic equilibrium (and thus to admit a temperature), we are looking for a temperature associated with black-body radiation and applicable to an object under the action of a force. This is of course the Unruh effect (Fulling [1973], Davies [1975], Unruh [1976]), whose characteristic temperature is given by \( T = (\hbar a)/(2\pi k_B c) \) and applicable to an object undergoing acceleration \( F := ma \). Making the replacements into \( F/T \), we get both the ratio \( F/T \) and \( T \) to remain uniform as required, and we obtain:

\[
dS = 2\pi k_B \frac{mc}{\hbar} d\bar{r}
\]  

(20)

where \( \hbar/(mc) \) is the well-known reduced Compton wavelength, here acting as the factor of proportionality between the entropy and the distance (Verlinde [2011]). We notice that the higher the mass, the higher the entropy. Since the most massive object for a given \( \bar{r} \) (radius) is a black hole, we pose \( \bar{r} := 2Gm/c^2 \) (the Schwarzschild radius) as the upper limit on entropy. The Black Hole also has the benefit of being isotropic, consistent with our assumptions in section (2.1). After integrating, we get \( S = 4\pi k_B Gm^2/(\hbar c) + C \), where \( C \) is an integration constant. Then, by posing \( A := 4\pi \bar{r}_s^2 = 16\pi G^2 m^2/c^4 \) and using the Planck length \( L_p := \sqrt{\hbar G/c^3} \), get a boundary on the size of the entropy as:

\[
S \leq k_B \frac{A}{4L_p^2} + C
\]  

(21)

which is proportional to the surface \( A \) and includes the factor 1/4. This result serves as a sanity check on the derivation so far: The
Bekenstein-Hawking entropy is recovered! Naturally, we interpret the surface as an event horizon and the ratio of temperature $\frac{r_0}{t_0}$ as the speed of light.

This is the result when we ignore the effect of time ($d\tilde{t} = 0$) on the entropy of the global system. What happens when we don’t?

3.4 Recording the passage of time

We integrate equation (14). We get:

$$\int dS = -\frac{k_B}{t_0} \int d\tilde{t} + \frac{k_B}{r_0} \int d\tilde{r}$$

$$\Delta S = -\frac{k_B}{t_0} \Delta \tilde{t} + \frac{k_B}{r_0} \Delta \tilde{r} + C$$

This relation leads to two inequalities:

$$0 \leq -\Delta S \leq \frac{k_B}{t_0} \Delta \tilde{t}$$
$$0 \leq \Delta S \leq \frac{k_B}{r_0} \Delta \tilde{r}$$

The first, involving time, relates the minimum amount of information ($-\Delta S$) that must be acquired to prove that time has passed by a certain amount $\Delta \tilde{t}$. It is interpreted in the sense that the passage of time $\Delta \tilde{t}$ requires the logging of an event $-\Delta S$. The second, involving space, relates an entropy ($\Delta S$) to the minimum increase in the size of space $\Delta \tilde{r}$ required to accommodate it. Let us now study this behavior in more detail by constructing an (abstract) thermodynamic engine that converts time to space.

3.5 Space-time engine

We will define a thermodynamic engine that converts time to space. The engine is comprised of a detector and a tape. The engine can write one bit on the tape at $r = 0$. It can also shift the tape to the right by one increment $\Delta \tilde{r}$ (in preparation to write the next bit). For purposes of idealization, we consider that the engine never runs out of tape. Finally, and without loss of generality, it helps for the purposes of the illustration to consider the more familiar case where the system is at thermodynamic equilibrium. Thus, we pose thermodynamic equilibrium to the inequalities with these replacements: $k_B/t_0 := P/T$ where $P$ is a power in [Joules/seconds], and $k_B/r_0 := F/T$ where $F$ is a force in [Joules/meters], and where $T$ is a temperature in [Kelvins]. We get:

$$0 \leq -\Delta S \leq \frac{P}{T} \Delta \tilde{t}$$
$$0 \leq \Delta S \leq \frac{F}{T} \Delta \tilde{r}$$
To ease the abstraction, we can picture a concrete system sharing similar characteristics (with some limitations), such as a seismograph tracing seismic data (collected over time) on a rolling tape (stored in space). The cycles of the engine occur in parallel and are completed over a time period $\Delta t$. Each cycle represents a logical step in the process (not chronological). They are:

1. Shifting the tape to the right by $\Delta r$ is favored by entropy as it increases it by $\Delta S$. Thus, an entropic force $F$ emerges which pulls on the tape. For this engine, we associate this increase in entropy to an undefined memory address ($\Delta S = k_B \ln 2$) which becomes available at position 0 of the tape when it is shifted by $\Delta r$.

2. The detector clicks (it produces information). To register the click, $\Delta I$ increases as $\Delta S$ is decreased. To pay the energy cost to decrease the entropy, the detector draws a power $P$ from the engine over the time period $\Delta t$.

3. Finally, the engine writes the bit associated with the click in the undefined memory address of the tape. This reclaims the energy of the shift ($F \Delta r$) that produced the entropy and instead makes it available to power the detector ($P \Delta I$).

This engine has the interesting property that the further along the tape one looks, the further back in time the information the bits on the tape refers to. At tempo-dynamic equilibrium, the tape is shifted at the speed of light towards the right, and its farthest bit refers to the very first thing that has been recorded.

4 Discussion

4.1 Three regimes of time

Reprising our discussion over the system’s qualitatively different description of past, present, and future, we now study the entropy dynamics of both time and distance. As concluded in the previous section, an observer perceives the "distance-only-entropy" ($dI = 0$) to be $S \leq k_B A / (4L^2)$. However, the situation is different at tempo-dynamic equilibrium. Consider the case when $dI$ also varies, given by equation (20), for which the ratio of its tempo-ture is $c$. If we were to repeat the steps of the proof of the previous section related to the maximum entropy, we would find that in the case of (20), the entropy reduction of increasing the $I$ quantity exactly compensates the entropy increase associated with increasing the $r$ quantity. From this, we would find that the observer, at tempo-dynamic equilibrium, always sees an entropy of 0 in his present (plus an integration constant
on the arrow of spacetime

— to be neglected from the discussion). The author realizes that this is quite the surprise, but as we will soon see, there are insights to be had.

Discussion:

First, please note that $S(t \equiv \text{now}) = 0$ occurs because we explicitly added the time quantity to statistical physics. Without this quantity, we are indeed lead to believe that the entropy of the present is very high. But "hold on", you say, is it not common knowledge that the entropy of the universe is very high, thus how are we getting $S = 0$ for the present instead of some very high number — what is going on here?

To interpret this result, we must consider that the entropy attributable to the different quantities $(\mathcal{f}, \mathcal{t})$ serves different purposes. Space-like entropy ($d\mathcal{t} = 0$) is found in equation (20) and is associated to well-known systems, such as the number of possible states the molecules of air in a room could be in. This boundary on this entropy is given by the equation without the time quantity and is estimated to be in the order of $\leq 10^{122}$ in the universe (using $S \leq k_B A / (4L_{\text{p}} ^2)$). Another kind, light-like entropy ($d\mathcal{r} \neq 0$ and $d\mathcal{t} \neq 0$), is given by equation (14). This entropy refers to the observer’s "present-experience" and reflects the number of states the observer itself can be in while still experiencing the same present. Equation (14) tells us this entropy is 0 as the present is defined by a single state. For instance, the observer does not experience a "superposition" of possible presents, and consequently, we expect the observer to "measure" Schrödinger’s cat as either dead or alive, but not as a superposition of both.

We associate space-like entropy to a trace of the past. As the waterfall recedes in space-time, it leaves a trace of events in the degrees of freedom of space. An observer can, by inspecting the trace, find evidence for a consistent past to account for the present. The horizon produced by the depletion of past events under Fermi-Dirac statistics prohibits $O$ from going backward or interacting with past events (Figure 2) directly. A similar horizon, also enforced by state depletion, is also found at the edge of the event horizon in space (Figure 1). The entropy in the degrees of freedom of space limits the uniqueness of the reconstruction of the past achievable by $O$ based on the analysis of the trace, as it represents the number of logs of events that are compatible with the observer’s unique present.

Finally, we associate time-like entropy to the future. The observer is prohibited from peeking into its future (increasing $\mathcal{t}$ above temporal equilibrium) until it is imminent. Indeed, an observer peeking into the future (without first waiting for the waterfall to recede appropriately) will hit negative entropy (contradiction). The observer
can hypothesize about its possible futures, but the actual future is made final no sooner than in the present when the entropy hits 0. This negative entropy prevents the macroscopic system from growing faster than allowed by tempo-dynamic equilibrium.

What about the arrow of time? This now accounts for only half of the story. The waterfall of events, as it recedes in space-time, creates two arrows related to time. The first arrow is the familiar one. The entropy associated with the degrees of freedom of space \(((k_B/r_0)d\bar{r})\) increases as time moves forward. The second arrow acts in the reverse direction on the degrees of freedom of time \((-k_B/t_0)dt\). With it, the entropy associated with time decreases as time moves forward because future possibilities are consumed to create a present.

4.2 Falsifiable prediction

From equation (14), and preserving the units, we impose thermodynamic equilibrium on the system with the replacement \(k_B/t_0 := P/T\) where \(P\) is a power [Joules/seconds] and \(T\) is a temperature [Kelvin]. By further posing \(d\bar{r} = 0\). We get:

\[
TdS = -Pd\bar{t}
\]  

(26)

This equation predicts that an entropic power can be made to emerge if information is produced (or consumed) over time, and at constant temperature.

4.3 Note on the signs of the Lagrange multipliers

The signs of the Lagrange multipliers were chosen for these reasons: 1) Starting with (14) and posing \(dS = 0\) we get \(d\bar{r} = cd\bar{t}\) (where \(c := r_0/t_0\)). This is the fundamental relation of special relativity connecting space to time. Thus, the signs of the Lagrange multipliers must be opposite. 2) In the derivation of the Bekenstein-Hawking entropy, we have used \(F = ma\), and not \(F = -ma\). Thus, the Lagrange multiplier \(-1/r_0\) must be negative. Finally, 1) and 2) implies that the sign of \(1/t_0\) must be positive.

5 Conclusion

We conclude that the statistical physics of space-time events admits:

1. The speed of light as the ratio of the tempo-ture of the system.
2. A waterfall of events receding towards the future and flooding the present with events.
3. A surface boundary to maintain the average growth of the system consistent with the speed of light and causality. This limits the entropy of the inner system proportionally to its area (Bekenstein-Hawking entropy).

4. A description of all three regimes of time (past, present, and future) distinctly from one another, and in a manner consistent with our macroscopic experience of said modes. Specifically;

(a) The observer perceives his macroscopic present with an entropy of 0, negating the possibility of being in a superposition of multiple possible presents,

(b) The observer cannot peak into its future without hitting negative entropy (contradiction),

(c) The observer cannot observe past events as their occupancy is depleted. At best, an attempt to reconstruct the past can be made based on a forensic analysis of the present. This reconstruction is not uniquely determined as multiple logs of events lead to the same present.

References


H-Dieter Zeh and Don N Page. The physical basis of the direction of time, 1990.