

# Question 475 : Pi , Fractals and Formulas

Edgar Valdebenito

23 August , 2018

abstract

In this note we give some fractals and formulas for Pi:

$$\pi = 3.1415926535\dots$$

## Introducción

En esta nota mostramos algunos fractales relacionados con la función:

$$F(x, y) = y^2 \sinh^4 x - 2y \sinh^3 x + y^2 \sinh^2 x - 4y \sinh x - y^2 + 1 \quad (1)$$

En particular consideramos los casos:  $y = 1/\sqrt{3}$ ,  $y = \sqrt{2} - 1$ ,  $y = 2 - \sqrt{3}$ . además mostramos algunas fórmulas del tipo:

$$\pi = k \tan^{-1}(\sinh x) \pm k \tan^{-1}\left(\left(\tanh \frac{x}{2}\right)^2\right), k \in \mathbb{N}, x \in \mathbb{R} \quad (2)$$

## Fractales

- Caso 1:  $y = 1/\sqrt{3}$

$$F\left(x, \frac{1}{\sqrt{3}}\right) = \frac{1}{3} \sinh^4 x - \frac{2}{3} \sqrt{3} \sinh^3 x + \frac{1}{3} \sinh^2 x - \frac{4}{3} \sqrt{3} \sinh x + \frac{2}{3} \quad (3)$$

- Caso 2 :  $y = \sqrt{2} - 1$

$$F(x, \sqrt{2} - 1) = (3 - 2\sqrt{2}) \sinh^4 x - (2\sqrt{2} - 2) \sinh^3 x + (3 - 2\sqrt{2}) \sinh^2 x - (4\sqrt{2} - 4) \sinh x + 2\sqrt{2} - 2 \quad (4)$$

- Caso 3 :  $y = 2 - \sqrt{3}$

$$F(x, 2 - \sqrt{3}) = (7 - 4\sqrt{3}) \sinh^4 x - (4 - 2\sqrt{3}) \sinh^3 x + (7 - 4\sqrt{3}) \sinh^2 x - (8 - 4\sqrt{3}) \sinh x + 4\sqrt{3} - 6 \quad (5)$$

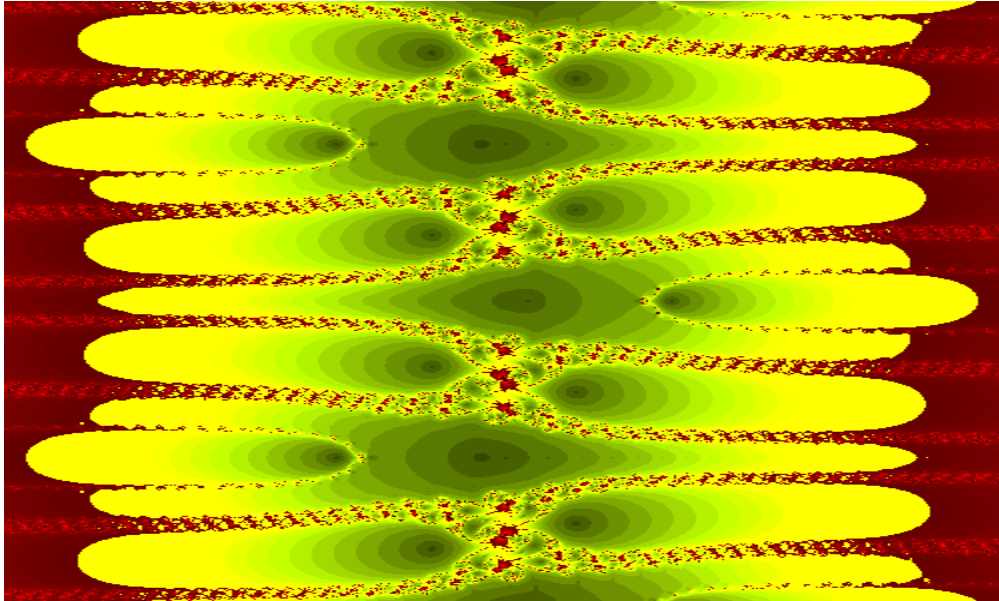


Figura 1.  $F(x, 1/\sqrt{3})$ ,  $x \in (-6-6i, 6+6i)$

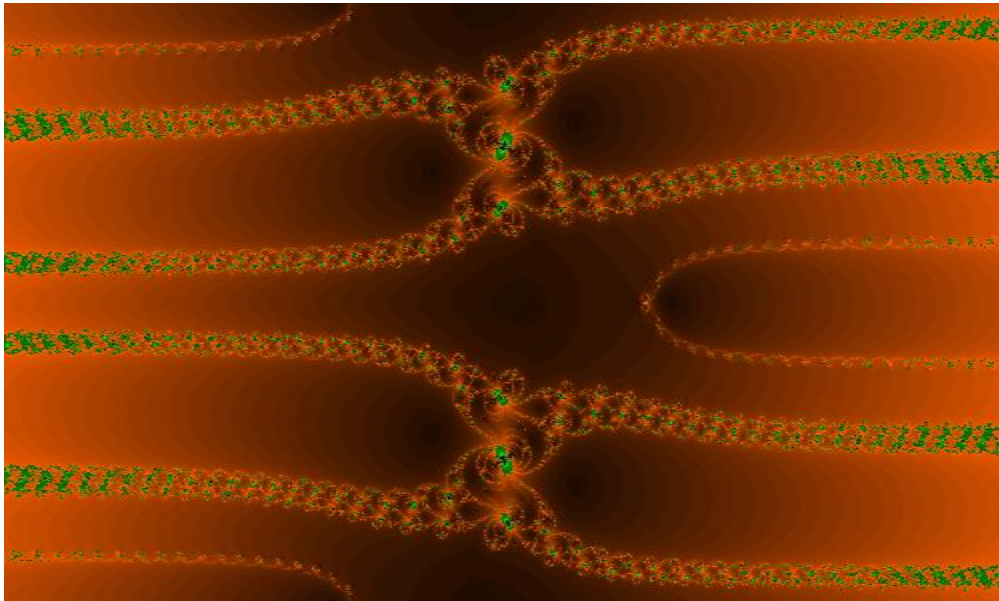


Figura 2.  $F(x, 1/\sqrt{3})$ ,  $x \in (-6-3i, 6+3i)$

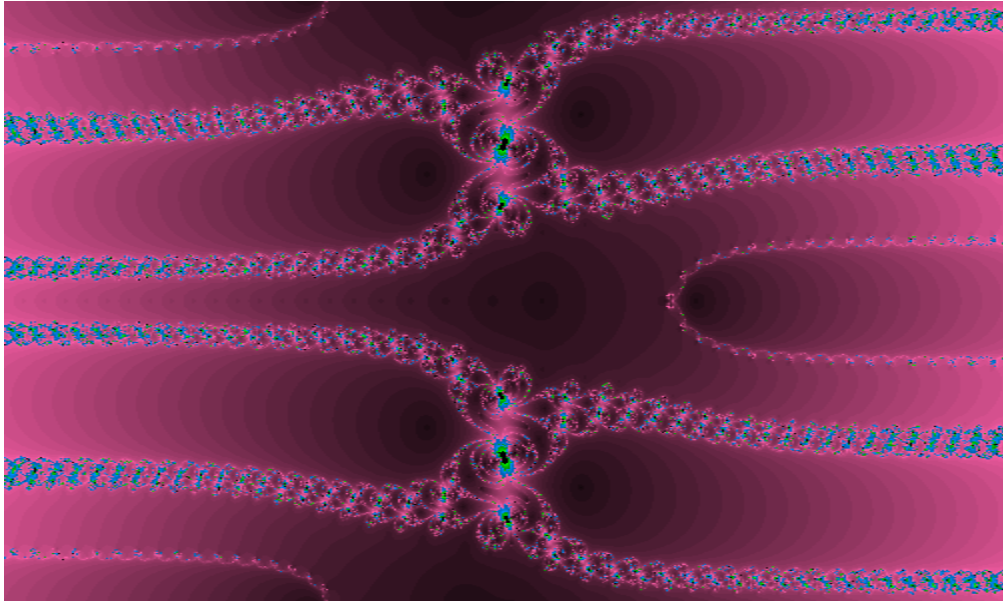


Figura 3.  $F(x, \sqrt{2}-1)$ ,  $x \in (-6-3i, 6+3i)$

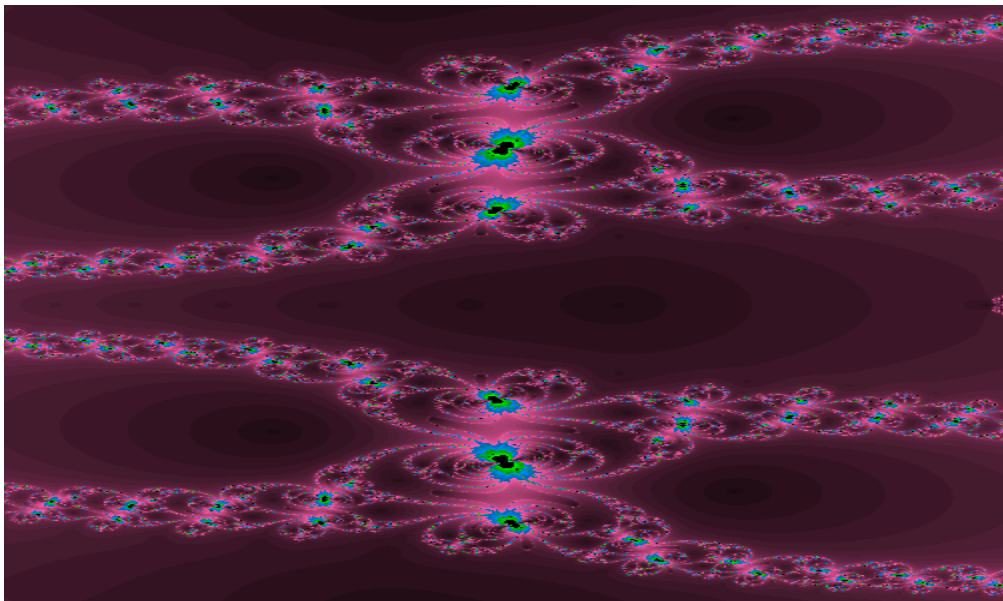


Figura 4.  $F(x, \sqrt{2}-1)$ ,  $x \in (-2-3i, 2+3i)$

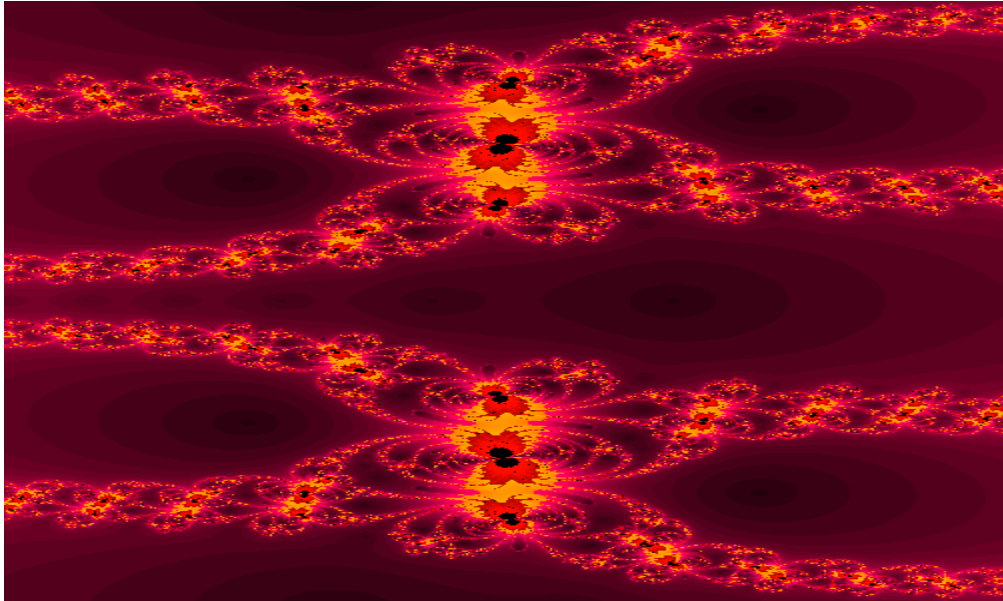


Figura 5.  $F(x, 2 - \sqrt{3})$ ,  $x \in (-2 - 3i, 2 + 3i)$

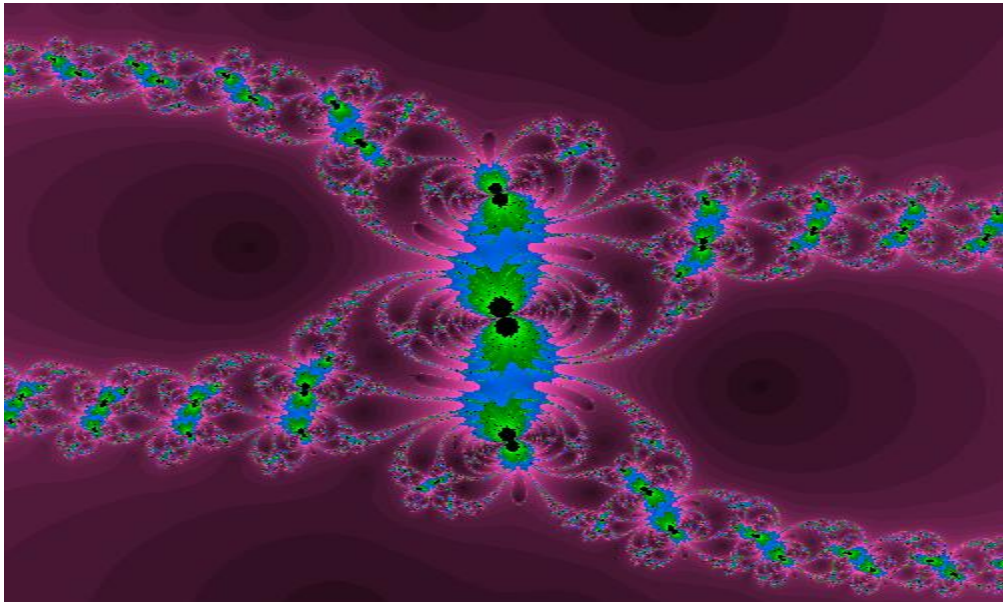


Figura 6.  $F(x, 2 - \sqrt{3})$ ,  $x \in (-2 + 0i, 2 + 3i)$

## Fórmulas Para Pi

- Caso 1:

$$F(x, 1/\sqrt{3}) = 0, x \in \mathbb{R} \Rightarrow x = x_1, x = x_2 \quad (6)$$

$$x_1 = \sinh u = \text{RootOf}(z^4 - 2\sqrt{3}z^3 + z^2 - 4\sqrt{3}z + 2, \text{index} = 1) = 0.2896... \quad (7)$$

$$x_2 = \sinh v = \text{RootOf}(z^4 - 2\sqrt{3}z^3 + z^2 - 4\sqrt{3}z + 2, \text{index} = 2) = 3.6662... \quad (8)$$

$$\pi = 12 \tan^{-1}(\sinh u) - 12 \tan^{-1}\left(\left(\tanh \frac{u}{2}\right)^2\right) \quad (9)$$

$$\pi = \frac{12}{7} \tan^{-1}(\sinh v) + \frac{12}{7} \tan^{-1}\left(\left(\tanh \frac{v}{2}\right)^2\right) \quad (10)$$

$$\pi = 12 \tan^{-1}\left(\left(\tanh \frac{v}{2}\right)^2\right) - 12 \tan^{-1}\left(\frac{1}{\sinh v}\right) \quad (11)$$

$$\pi = 6 \tan^{-1}\left(\frac{1}{\cosh v}\right) + 6 \tan^{-1}\left(\frac{1}{\sinh v}\right) \quad (12)$$

- Caso 2:

$$F(x, \sqrt{2}-1) = 0, x \in \mathbb{R} \Rightarrow x = x_1, x = x_2 \quad (13)$$

$$x_1 = \sinh u = \text{RootOf}(z^4 - (2+2\sqrt{2})z^3 + z^2 - (4+4\sqrt{2})z + 2+2\sqrt{2}, \text{index} = 1) \quad (14)$$

$$x_1 = 0.4750... \quad (15)$$

$$x_2 = \sinh v = \text{RootOf}(z^4 - (2+2\sqrt{2})z^3 + z^2 - (4+4\sqrt{2})z + 2+2\sqrt{2}, \text{index} = 2) \quad (16)$$

$$x_2 = 4.9780... \quad (17)$$

$$\pi = 8 \tan^{-1}(\sinh u) - 8 \tan^{-1}\left(\left(\tanh \frac{u}{2}\right)^2\right) \quad (18)$$

$$\pi = \frac{8}{5} \tan^{-1}(\sinh v) + \frac{8}{5} \tan^{-1}\left(\left(\tanh \frac{v}{2}\right)^2\right) \quad (19)$$

$$\pi = 8 \tan^{-1}\left(\left(\tanh \frac{v}{2}\right)^2\right) - 8 \tan^{-1}\left(\frac{1}{\sinh v}\right) \quad (20)$$

$$\pi = 8 \tan^{-1} \left( \frac{1}{\cosh v} \right) + 8 \tan^{-1} \left( \frac{1}{\sinh v} \right) \quad (21)$$

- Caso 3:

$$F(x, 2 - \sqrt{3}) = 0, x \in \mathbb{R} \Rightarrow x = x_1, x = x_2 \quad (22)$$

$$x_1 = \sinh u = \text{RootOf} \left( z^4 - (4 + 2\sqrt{3})z^3 + z^2 - (8 + 4\sqrt{3})z + 6 + 4\sqrt{3}, \text{index} = 1 \right) \quad (23)$$

$$x_1 = 0.7276... \quad (24)$$

$$x_2 = \sinh v = \text{RootOf} \left( z^4 - (4 + 2\sqrt{3})z^3 + z^2 - (8 + 4\sqrt{3})z + 6 + 4\sqrt{3}, \text{index} = 2 \right) \quad (25)$$

$$x_2 = 7.5629... \quad (26)$$

$$\pi = 6 \tan^{-1}(\sinh u) - 6 \tan^{-1} \left( \left( \tanh \frac{u}{2} \right)^2 \right) \quad (27)$$

$$\pi = \frac{3}{2} \tan^{-1}(\sinh v) + \frac{3}{2} \tan^{-1} \left( \left( \tanh \frac{v}{2} \right)^2 \right) \quad (28)$$

$$\pi = 6 \tan^{-1} \left( \left( \tanh \frac{v}{2} \right)^2 \right) - 6 \tan^{-1} \left( \frac{1}{\sinh v} \right) \quad (29)$$

$$\pi = 6 \tan^{-1} \left( \frac{1}{\cosh v} \right) + 6 \tan^{-1} \left( \frac{1}{\sinh v} \right) \quad (30)$$

Obsevación: La notación RootOf es la utilizada en Mathematica o Maple.

## Referencias

1. Barnsley, M.F. and Rising, H.: Fractals Everywhere, 2nd ed. Boston, MA : Academic Press, 1993.
2. Beckmann, P.: A History of Pi, 3rd ed. New York: Dorset Press, 1989.
3. Blatner, D.: The Joy of Pi. New York: Walker, 1997.
4. Bunde, A. and Havlin, S. (Eds.): Fractals in Science. New York: Spriger-Verlag, 1994.
5. Devaney, R.L.: Complex Dynamical Systems: The Mathematics Behind the Mandelbrot and Julia Sets. Providence, RI: Amer. Math. Soc., 1994.
6. Valdebenito, E.: Question 201: A fractal image. <http://vixra.org/pdf/1710.0131v1pdf> .
7. Valdebenito, E.: Ramanujan Trigonometric Formula. <http://vixra.org/pdf/1707.0241v1pdf> .