Some Relations for the q-Pochhammer Symbol, the q-Binomial Coefficient, the q-Bracket, the q-Factorial and the q-Gamma Function

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I deduce some relations for the q-Pochhammer symbol, the q-binomial coefficient, q-bracket, q-factorial and q-Gamma function.

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1. INTRODUCTION

In this paper, I proved the recurrence relation for the q-Pochhammer symbol:

$$(a;q)_k = (1+q-aq^{k-1})(a;q)_{k-1} - (1-aq^{k-2})q(a;q)_{k-2};$$

the recurrence relation for the q-binomial coefficient:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{1+q-q^n}{1-q^m}\cdot\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q - \frac{(1-q^{n-1})q}{(1-q^{m-1})(1-q^m)}\cdot\begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q;$$

the relation for the q-bracket:

$$\frac{q[n]_q[n+1]_q}{1+q} = (1+q-q^{n+2})\cdot\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - (1-q^3)\cdot\begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q;$$

the recurrence relation for the q-factorial:

$$(1-q)[k]_q! = (1+q-q^k)[k-1]_q! - \left(\frac{1-q^{k-1}}{1-q}\right)q[k-2]_q!;$$

and the recurrence relation for the q-Gamma function:

$$(1-q^k)\Gamma_q(k) = (1+q-q^k)\left(\frac{1-q^{k-1}}{1-q}\right)\Gamma_q(k-1)$$
$$-(1-q^{k-1})\left[\frac{1-q^{k-2}}{(1-q)^2}\right]q\Gamma_q(k-2).$$

2. THE RECURRENCE RELATION FOR THE q-POCHHAMMER SYMBOL

Theorem 2.1. *If* $|a|<1$, $|q|<1$ *and* $k\geqslant 0$, *then*

$$(a;q)_k=(1+q-aq^{k-1})(a;q)_{k-1}-(1-aq^{k-2})q(a;q)_{k-2}, \tag{2.1}$$

where $(a;q)_k$ *denotes the q-Pochhammer symbol, defined [1] by*

$$(a;q)_k:=\begin{cases}\prod_{j=0}^{k-1}(1-aq^j), \text{if } k>0\\ 1, \text{if } k=0\\ \prod_{j=1}^{|k|}(1-aq^{-j})^{-1}, \text{if } k<0\\ \prod_{j=0}^{\infty}(1-aq^j), \text{if } k=\infty.\end{cases} \tag{2.2}$$

Proof. Step 1. I consider the finite product representation for the q-Pochhammer symbol

$$(a;q)_k=\prod_{j=0}^{k-1}(1-aq^j), \tag{2.3}$$

provided $k>0$.

I suppose that

$$\alpha\cdot(a;q)_{k+2}=\beta\cdot(a;q)_{k+1}+\gamma\cdot(a;q)_k. \tag{2.4}$$

Substituting the right hand side of (2.3) in (2.4), I obtain

$$\begin{gathered}\alpha\cdot\prod_{j=0}^{k+1}(1-aq^j)=\beta\cdot\prod_{j=0}^{k}(1-aq^j)+\gamma\cdot\prod_{j=0}^{k-1}(1-aq^j)\\ \Rightarrow\alpha\cdot\frac{\prod_{j=0}^{k+1}(1-aq^j)}{\prod_{j=0}^{k-1}(1-aq^j)}=\beta\cdot\frac{\prod_{j=0}^{k}(1-aq^j)}{\prod_{j=0}^{k-1}(1-aq^j)}+\gamma\\ \Rightarrow\alpha\cdot(1-aq^k)(1-aq^{k+1})=\beta\cdot(1-aq^k)+\gamma\\ \Rightarrow\alpha\cdot(1-aq^{k+1}-aq^k+a^2q^{2k+1})=\beta\cdot(1-aq^k)+\gamma\\ \Rightarrow\alpha-\alpha aq^{k+1}-\alpha aq^k+\alpha a^2q^{2k+1}=\beta+\gamma-\beta aq^k\\ \Rightarrow\alpha-(1+q-aq^{k+1})\alpha aq^k=\beta+\gamma-\beta aq^k.\end{gathered} \tag{2.5}$$

Comparing the coefficients in (2.5), I conclude that

$$\begin{cases}\alpha=\beta+\gamma\\ (1+q-aq^{k+1})\alpha=\beta\end{cases}\sim\begin{cases}\gamma=-(1-aq^k)\alpha q\\ \beta=(1+q-aq^{k+1})\alpha.\end{cases} \tag{2.6}$$

Replace β by $(1+q-aq^{k+1})\alpha$ and γ by $-(1-aq^k)\alpha q$ in (2.4) and encounter

$$\begin{gathered}\alpha\cdot(a;q)_{k+2}=(1+q-aq^{k+1})\alpha\cdot(a;q)_{k+1}-(1-aq^k)\alpha q\cdot(a;q)_k\\ \Rightarrow(a;q)_{k+2}=(1+q-aq^{k+1})(a;q)_{k+1}-(1-aq^k)q(a;q)_k.\end{gathered} \tag{2.7}$$

Replace k by $k-2$ in both members of (2.7)

$$(a;q)_k=(1+q-aq^{k-1})(a;q)_{k-1}-(1-aq^{k-2})q(a;q)_{k-2}, \tag{2.8}$$

which is valid for $k\geqslant 3$.

Step 2. I prove (2.8) for $k=0$. Set $k=0$ in (2.8)

$$(a;q)_0=(1+q-aq^{-1})(a;q)_{-1}-(1-aq^{-2})q(a;q)_{-2}, \tag{2.9}$$

By (2.2), I have

$$(a;q)_0=1, \tag{2.10}$$

$$(a;q)_{-1}=\frac{1}{1-\frac{a}{q}}. \tag{2.11}$$

and

$$(a;q)_{-2}=\frac{1}{\left(1-\frac{a}{q}\right)\left(1-\frac{a}{q^2}\right)}. \tag{2.12}$$

From (2.9) at (2.12), it follows that

$$1=\left(1+q-\frac{a}{q}\right)\frac{1}{1-\frac{a}{q}}-\left(1-\frac{a}{q^2}\right)q\frac{1}{\left(1-\frac{a}{q}\right)\left(1-\frac{a}{q^2}\right)}. \tag{2.13}$$

The expansion for the right hand side of (2.13) give me

$$\begin{aligned}
&\left(1+q-\frac{a}{q}\right)\frac{1}{1-\frac{a}{q}}-\left(1-\frac{a}{q^2}\right)q\frac{1}{\left(1-\frac{a}{q}\right)\left(1-\frac{a}{q^2}\right)}\\
&=\left(1+q-\frac{a}{q}\right)\frac{1}{1-\frac{a}{q}}-q\frac{1}{1-\frac{a}{q}}\\
&=\left(\frac{q+q^2-a}{q}\right)\frac{q}{q-a}-q\frac{q}{q-a}\\
&=\frac{q+q^2-a}{q-a}-\frac{q^2}{q-a}\\
&=\frac{q+q^2-a-q^2}{q-a}\\
&=\frac{q-a}{q-a}=1.
\end{aligned} \tag{2.14}$$

Note that the right hand side of (2.14) is equal to the left hand side of (2.13). This completes the proof for $k=0$.

Step 3. I prove (2.8) for $k=1$. Set $k=1$ in (2.8)

$$(a;q)_1=(1+q-a)(a;q)_0-(1-aq^{-1})q(a;q)_{-1}. \tag{2.15}$$

By (2.2), I have

$$(a;q)_1=1-a, \tag{2.16}$$

$$(a;q)_0=1 \tag{2.17}$$

and

$$(a;q)_{-1}=\frac{1}{1-\frac{a}{q}}. \tag{2.18}$$

From (2.15) at (2.18), it follows that

$$1-a=(1+q-a)-(1-aq^{-1})q\frac{1}{1-\frac{a}{q}}. \tag{2.19}$$

The expansion for the right hand side of (2.19) give me

$$\begin{aligned}&(1+q-a)-(1-aq^{-1})q\frac{1}{1-\frac{a}{q}}\\&=1+q-a-\left(\frac{q-a}{q}\right)q\frac{q}{q-a}\\&=1+q-a-q\\&=1-a.\end{aligned} \tag{2.20}$$

Note that the right hand side of (2.20) is equal to the left hand side of (2.19). This completes the proof for $k=1$.

Step 4. I prove (2.8) for $k=2$. Set $k=2$ in (2.8)

$$(a;q)_2=(1+q-aq)(a;q)_1-(1-a)q(a;q)_0. \tag{2.21}$$

By (2.2), I have

$$(a;q)_2=(1-a)(1-aq), \tag{2.22}$$

$$(a;q)_1=1-a \tag{2.23}$$

and

$$(a;q)_0=1. \tag{2.24}$$

From (2.21) at (2.24), it follows that

$$(1-a)(1-aq)=(1+q-aq)(1-a)-(1-a)q. \tag{2.25}$$

The expansion for the right hand side of (2.25) give me

$$\begin{aligned}&(1+q-aq)(1-a)-(1-a)q\\&=(1+q-aq-q)(1-a)\\&=(1-a)(1-aq).\end{aligned} \tag{2.26}$$

Note that the right hand side of (2.26) is equal to the left hand side of (2.25). This completes the proof for $k=2$.

Step 4 (Conclusion). With the step 1, I proved that (2.8) is valid for $k\geqslant 3$; with the step 2, I proved that (2.8) is valid for $k=0$; with the step 3, I proved that (2.8) is valid for $k=1$; with the step 4, I proved that (2.8) is valid for $k=2$. Thus, I completed the proof that the equation (2.8) is valid for $k\geqslant 0$, which is the desired result. □

Remark 2.2. Obviously, when $k\to\infty$, the equation (2.1) becomes

$$(a;q)_\infty=\lim_{k\to\infty}(1+q-aq^{k-1})(a;q)_\infty-\lim_{k\to\infty}(1-aq^{k-2})q(a;q)_\infty. \tag{2.27}$$

Note that

$$\lim_{k\to\infty}(1+q-aq^{k-1})=1+q-\lim_{k\to\infty}(aq^{k-1})=1+q, \tag{2.28}$$

since $\lim\limits_{k\to\infty} q^{k-1}=0$, because $|q|<1$.

Notice also that

$$\lim_{k\to\infty}(1-aq^{k-2})=1-\lim_{k\to\infty}(aq^{k-2})=1, \tag{2.29}$$

since $\lim\limits_{k\to\infty} q^{k-2}=0$, because $|q|<1$.

From (2.27) at (2.29), I conclude that

$$(a;q)_\infty=(1+q)(a;q)_\infty-q(a;q)_\infty. \tag{2.30}$$

The expansion of the right hand side of (2.30) give me

$$(1+q)(a;q)_\infty-q(a;q)_\infty=(a;q)_\infty+q(a;q)_\infty-q(a;q)_\infty=(a;q)_\infty. \tag{2.31}$$

Observe that the right hand side of (2.31) is equal to the left hand side of (2.30) and this completes the proof, when $k\to\infty$.

Corollary 2.3. *If* $|a|<1$, $|q|<1$ *and* $k\geqslant 0$, *then*

$$\frac{(a;q)_{k+2}-(a;q)_{k+1}}{(a;q)_{k+1}-(a;q)_k}=q(1-aq^k), \tag{2.32}$$

where $(a;q)_k$ *denotes the q-Pochhammer symbol, defined [1] by*

$$(a;q)_k:=\begin{cases}\prod_{j=0}^{k-1}(1-aq^j), \text{if } k>0\\ 1, \text{if } k=0\\ \prod_{j=1}^{|k|}(1-aq^{-j})^{-1}, \text{if } k<0\\ \prod_{j=0}^{\infty}(1-aq^j), \text{if } k=\infty.\end{cases} \tag{2.33}$$

Proof. Expand the right-hand side of the Theorem 2.1 as follows:

$$\begin{aligned}&(1+q-aq^{k-1})(a;q)_{k-1}-(1-aq^{k-2})q(a;q)_{k-2}\\ =&[1+q(1-aq^{k-2})](a;q)_{k-1}-(1-aq^{k-2})q(a;q)_{k-2}\\ =&(a;q)_{k-1}+(1-aq^{k-2})q(a;q)_{k-1}-(1-aq^{k-2})q(a;q)_{k-2}\\ =&(a;q)_{k-1}+[(a;q)_{k-1}-(a;q)_{k-2}]q(1-aq^{k-2}).\end{aligned} \tag{2.34}$$

From (2.1) and (2.34), I conclude that

$$\begin{aligned}&(a;q)_k=(a;q)_{k-1}+[(a;q)_{k-1}-(a;q)_{k-2}]q(1-aq^{k-2})\\ \Rightarrow&(a;q)_k-(a;q)_{k-1}=[(a;q)_{k-1}-(a;q)_{k-2}]q(1-aq^{k-2})\\ \Rightarrow&\frac{(a;q)_k-(a;q)_{k-1}}{(a;q)_{k-1}-(a;q)_{k-2}}=q(1-aq^{k-2}).\end{aligned} \tag{2.35}$$

Replace k by $k+2$ in (2.35)

$$\frac{(a;q)_{k+2}-(a;q)_{k+1}}{(a;q)_{k+1}-(a;q)_k}=q(1-aq^k),$$

which is the desired result. □

3. THE RECURRENCE RELATION FOR THE q-BINOMIAL COEFFICIENT

Theorem 3.1. *If* $|q|<1$ *and* $2\leqslant m\leqslant n$*, then*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q=\frac{1+q-q^n}{1-q^m}\cdot\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q-\frac{(1-q^{n-1})q}{(1-q^{m-1})(1-q^m)}\cdot\begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q, \tag{3.1}$$

where $\begin{bmatrix} n \\ m \end{bmatrix}_q$ *denotes the q-binomial coefficient, defined [2, (1)] by*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q:=\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}. \tag{3.2}$$

Proof. Replace a by q and k by n in (2.1)

$$(q;q)_n=(1+q-q^n)(q;q)_{n-1}-(1-q^{n-1})q(q;q)_{n-2}. \tag{3.3}$$

Divide both members of (3.3) by $(q;q)_m(q;q)_{n-m}$ and encounter

$$\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}=(1+q-q^n)\frac{(q;q)_{n-1}}{(q;q)_m(q;q)_{n-m}}-(1-q^{n-1})q\frac{(q;q)_{n-2}}{(q;q)_m(q;q)_{n-m}}. \tag{3.4}$$

The manipulation of the right hand side of (3.4) give me

$$\begin{aligned}
&(1+q-q^n)\frac{(q;q)_{n-1}}{(q;q)_m(q;q)_{n-m}}-(1-q^{n-1})q\frac{(q;q)_{n-2}}{(q;q)_m(q;q)_{n-m}}\\
=&(1+q-q^n)\frac{(q;q)_{m-1}(q;q)_{n-1}}{(q;q)_m(q;q)_{m-1}(q;q)_{n-m}}\\
&-(1-q^{n-1})q\frac{(q;q)_{m-2}(q;q)_{n-2}}{(q;q)_m(q;q)_{m-2}(q;q)_{n-m}}\\
=&(1+q-q^n)\frac{(q;q)_{m-1}}{(q;q)_m}\cdot\frac{(q;q)_{n-1}}{(q;q)_{m-1}(q;q)_{n-m}}\\
&-(1-q^{n-1})q\frac{(q;q)_{m-2}}{(q;q)_m}\cdot\frac{(q;q)_{n-2}}{(q;q)_{m-2}(q;q)_{n-m}}\\
=&\frac{1+q-q^n}{1-q^m}\cdot\frac{(q;q)_{n-1}}{(q;q)_{m-1}(q;q)_{n-m}}\\
&-\frac{(1-q^{n-1})q}{(1-q^{m-1})(1-q^m)}\cdot\frac{(q;q)_{n-2}}{(q;q)_{m-2}(q;q)_{n-m}}.
\end{aligned} \tag{3.5}$$

From (3.2), (3.4) and (3.5), I conclude that

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{1+q-q^n}{1-q^m} \cdot \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q - \frac{(1-q^{n-1})q}{(1-q^{m-1})(1-q^m)} \cdot \begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q,$$

which is the desired result. □

4. A Relation for the q-Bracket

Theorem 4.1. *If* $|q|<1$ *and* $0 \leqslant n$, *then*

$$[n]_q = \frac{(1-q^2)(1+q-q^{n+2})}{(1-q^{n+1})q} \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - \frac{(1-q^2)(1-q^3)}{(1-q^{n+1})q} \cdot \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q, \tag{4.1}$$

where $[n]_q$ *denotes the q-bracket, defined [2, (5)] by*

$$[n]_q := \begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{1-q^n}{1-q}, \tag{4.2}$$

here $\begin{bmatrix} n \\ m \end{bmatrix}_q$ *denotes the q-binomial coefficient, defined as (3.2).*

Proof. Replace n by $n+2$ and m by $m+2$ in (3.1)

$$\begin{bmatrix} n+2 \\ m+2 \end{bmatrix}_q = \frac{1+q-q^{n+2}}{1-q^{m+2}} \cdot \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q - \frac{(1-q^{n+1})q}{(1-q^{m+1})(1-q^{m+2})} \cdot \begin{bmatrix} n \\ m \end{bmatrix}_q, \tag{4.3}$$

Replace m by 1 in (4.3)

$$\begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q = \frac{1+q-q^{n+2}}{1-q^3} \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - \frac{(1-q^{n+1})q}{(1-q^2)(1-q^3)} \cdot \begin{bmatrix} n \\ 1 \end{bmatrix}_q. \tag{4.4}$$

From (4.2) and (4.4), I obtain

$$[n]_q = \frac{(1-q^2)(1+q-q^{n+2})}{(1-q^{n+1})q} \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - \frac{(1-q^2)(1-q^3)}{(1-q^{n+1})q} \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q,$$

which is the desired result. □

Corollary 4.2. *If* $|q|<1$ *and* $0 \leqslant n$, *then*

$$\frac{q[n]_q[n+1]_q}{1+q} = (1+q-q^{n+2}) \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - (1-q^3) \cdot \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q, \tag{4.5}$$

where $[n]_q$ *denotes the q-bracket, defined [2, (5)] by*

$$[n]_q := \begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{1-q^n}{1-q}, \tag{4.6}$$

here $\begin{bmatrix} n \\ m \end{bmatrix}_q$ *denotes the q-binomial coefficient, defined as (3.2).*

Proof. Replace n by $n+1$ in (4.2)

$$[n+1]_q := \begin{bmatrix} n+1 \\ q \end{bmatrix}_q = \frac{1-q^{n+1}}{1-q}. \tag{4.7}$$

The manipulation of (4.1) give me

$$\begin{aligned} [n]_q = & \frac{1-q}{1-q^{n+1}} \cdot \frac{1+q}{q} \cdot (1+q-q^{n+2}) \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q \\ & - \frac{1-q}{1-q^{n+1}} \cdot \frac{1+q}{q} \cdot (1-q^3) \cdot \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q . \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), I conclude that

$$\begin{aligned} & [n]_q = \frac{1+q}{q[n+1]_q} \cdot (1+q-q^{n+2}) \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - \frac{1+q}{q[n+1]_q} \cdot (1-q^3) \cdot \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q \\ & \Rightarrow \frac{q[n]_q[n+1]_q}{1+q} = (1+q-q^{n+2}) \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - (1-q^3) \cdot \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q , \end{aligned}$$

which is the desired result. □

5. A RECURRENCE RELATION FOR THE q-FACTORIAL

Theorem 5.1. *If* $|q| < 1$ *and* $2 \leqslant k$*, then*

$$(1-q)[k]_q! = (1+q-q^k)[k-1]_q! - \left(\frac{1-q^{k-1}}{1-q}\right) q[k-2]_q!, \tag{5.1}$$

where $[k]_q!$ *denotes the q-factorial, defined [3] by*

$$[k]_q! := \frac{(q;q)_k}{(1-q)^k}. \tag{5.2}$$

Proof. Replace a by q in (2.1)

$$(q;q)_k = (1+q-q^k)(q;q)_{k-1} - (1-q^{k-1})q(q;q)_{k-2}. \tag{5.3}$$

Divide both members of (5.3) by $(1-q)^k$ and with a bit of manipulation obtain

$$\begin{aligned} & \frac{(q;q)_k}{(1-q)^k} = (1+q-q^k)\frac{(q;q)_{k-1}}{(1-q)^k} - (1-q^{k-1})q\frac{(q;q)_{k-2}}{(1-q)^k} \\ & \Rightarrow \frac{(q;q)_k}{(1-q)^k} = \left(\frac{1+q-q^k}{1-q}\right)\frac{(q;q)_{k-1}}{(1-q)^{k-1}} - \left[\frac{1-q^{k-1}}{(1-q)^2}\right] q \frac{(q;q)_{k-2}}{(1-q)^{k-2}}. \end{aligned} \tag{5.4}$$

Replace k by $k-1$ in (5.2)

$$[k-1]_q! = \frac{(q;q)_{k-1}}{(1-q)^{k-1}}. \tag{5.5}$$

Replace k by $k-2$ in (5.2)

$$[k-2]_q! = \frac{(q;q)_{k-2}}{(1-q)^{k-2}}. \tag{5.6}$$

From (5.2), (5.4), (5.5) and (5.6), I conclude that

$$[k]_q! = \left(\frac{1+q-q^k}{1-q}\right)[k-1]_q! - \left[\frac{1-q^{k-1}}{(1-q)^2}\right] q[k-2]_q!$$
$$\Rightarrow (1-q)[k]_q! = (1+q-q^k)[k-1]_q! - \left(\frac{1-q^{k-1}}{1-q}\right) q[k-2]_q!,$$

which is the desired result. □

6. A Recurrence Relation for the q-Gamma Function

Theorem 6.1. *If* $|q|<1$ *and* $3 \leqslant k$, *then*

$$\begin{aligned}(1-q^k)\Gamma_q(k) &= (1+q-q^k)\left(\frac{1-q^{k-1}}{1-q}\right)\Gamma_q(k-1)\\ &-(1-q^{k-1})\left[\frac{1-q^{k-2}}{(1-q)^2}\right] q\Gamma_q(k-2),\end{aligned} \tag{6.1}$$

where $\Gamma_q(k)$ *denotes the q-Gamma function, defined [4] by*

$$\Gamma_q(k) := \frac{(q;q)_\infty}{(q^k;q)_\infty}(1-q)^{1-k}. \tag{6.2}$$

Proof. In [5, p. 300, (12.1.3)], I have the identity

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{6.3}$$

Replace k by $k-1$ in (6.3)

$$(a;q)_{k-1} = \frac{(a;q)_\infty}{(aq^{k-1};q)_\infty}. \tag{6.4}$$

Replace k by $k-2$ in (6.3)

$$(a;q)_{k-2} = \frac{(a;q)_\infty}{(aq^{k-2};q)_\infty}. \tag{6.5}$$

From (2.1), (6.3), (6.4) and (6.5), it follows that

$$\frac{(a;q)_\infty}{(aq^k;q)_\infty} = (1+q-aq^{k-1})\frac{(a;q)_\infty}{(aq^{k-1};q)_\infty} - (1-aq^{k-2})q\frac{(a;q)_\infty}{(aq^{k-2};q)_\infty}. \tag{6.6}$$

Replace a by q in (6.6) and with a bit of manipulation obtain

$$\begin{aligned}
&\frac{(q;q)_\infty}{(q^{k+1};q)_\infty} = (1+q-q^k)\frac{(q;q)_\infty}{(q^k;q)_\infty} - (1-q^{k-1})q\frac{(q;q)_\infty}{(q^{k-1};q)_\infty} \\
&\Rightarrow \frac{(q;q)_\infty}{(q^k;q)_\infty}\cdot\frac{(q^k;q)_\infty}{(q^{k+1};q)_\infty} = (1+q-q^k)\frac{(q;q)_\infty}{(q^{k-1};q)_\infty}\cdot\frac{(q^{k-1};q)_\infty}{(q^k;q)_\infty} \\
&-(1-q^{k-1})q\frac{(q;q)_\infty}{(q^{k-2};q)_\infty}\cdot\frac{(q^{k-2};q)_\infty}{(q^{k-1};q)_\infty} \\
&\Rightarrow \frac{(q;q)_\infty}{(q^k;q)_\infty}\cdot\frac{(q;q)_k}{(q;q)_{k-1}} = (1+q-q^k)\frac{(q;q)_\infty}{(q^{k-1};q)_\infty}\cdot\frac{(q;q)_{k-1}}{(q;q)_{k-2}} \\
&-(1-q^{k-1})q\frac{(q;q)_\infty}{(q^{k-2};q)_\infty}\cdot\frac{(q;q)_{k-2}}{(q;q)_{k-3}} \\
&\Rightarrow (1-q^k)\frac{(q;q)_\infty}{(q^k;q)_\infty} = (1+q-q^k)(1-q^{k-1})\frac{(q;q)_\infty}{(q^{k-1};q)_\infty} \\
&-(1-q^{k-1})(1-q^{k-2})q\frac{(q;q)_\infty}{(q^{k-2};q)_\infty}.
\end{aligned} \tag{6.7}$$

Multiply both members of (6.7) by $(1-q)^{1-k}$ and with a bit of manipulation, I find

$$\begin{aligned}
&(1-q^k)\frac{(q;q)_\infty}{(q^k;q)_\infty}(1-q)^{1-k} = (1+q-q^k)\left(\frac{1-q^{k-1}}{1-q}\right)\frac{(q;q)_\infty}{(q^{k-1};q)_\infty}(1-q)^{2-k} \\
&-(1-q^{k-1})\frac{(1-q^{k-2})}{(1-q)^2}q\frac{(q;q)_\infty}{(q^{k-2};q)_\infty}(1-q)^{3-k}.
\end{aligned} \tag{6.8}$$

Replace k by $k-1$ in (6.2)

$$\Gamma_q(k-1) := \frac{(q;q)_\infty}{(q^{k-1};q)_\infty}(1-q)^{2-k}. \tag{6.9}$$

Replace k by $k-2$ in (6.2)

$$\Gamma_q(k-2) := \frac{(q;q)_\infty}{(q^{k-2};q)_\infty}(1-q)^{3-k}. \tag{6.10}$$

From (6.2), (6.8), (6.9) and (6.10), it follows that

$$\begin{aligned}
&(1-q^k)\Gamma_q(k) = (1+q-q^k)\left(\frac{1-q^{k-1}}{1-q}\right)\Gamma_q(k-1) \\
&-(1-q^{k-1})\left[\frac{1-q^{k-2}}{(1-q)^2}\right]q\Gamma_q(k-2),
\end{aligned}$$

which is the desired result. □

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