On The Non-Real Nature of $x \cdot 0$ ($x \in \mathbb{R}_{\neq 0}$)
The Set of Null-Imaginary Numbers

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Abstract

In this work I axiomatize the result of $x \cdot 0$ ($x \in \mathbb{R}_{\neq 0}$) as a number $i_0(x)$ that has a null real part (denoted as $\Re(i_0(x)) = 0$) but that is not real. This implies that $y + \Re(i_0(x)) = y$ but $y + i_0(x) = y + x \cdot 0 \neq y$, $y \in \mathbb{R}_{\neq 0}$. From this I define the set of null imaginary numbers $\mathbb{I}_\emptyset = \{i_0(x) = x \cdot 0 | \forall x \in \mathbb{R}_{\neq 0}\}$ and present its elementary algebra taking the axiom of uniqueness as base (i.e., if $x \neq y \Leftrightarrow i_0(x) \neq i_0(y)$). Under the condition of existence of $\mathbb{I}_\emptyset$ I show that division by zero can be defined without causing inconsistencies in elementary algebra.

1 Introduction

The Property of Multiplication by Zero (PMZ) defines zero as the 'absorbing element' on $\mathbb{R}$, i.e. $x \cdot 0 = 0$, $x \in \mathbb{R}_{\neq 0}$ [7, 6, 5]. This suggests the information of two *distinct* variables x and y are somehow 'lost' after multiplication by zero such that $x \cdot 0 = y \cdot 0$ does hold. The fact that PMZ seems to conflict with principles of natural sciences (e.g. conservation of mass, conservation of energy) is not a relevant concern for algebra. However, it is well known that PMZ does not coexist with its reciprocal in known algebraic structures. In fact, in elementary algebra PMZ is defined and its reciprocal is not [1]. In [1], for instance, the author proposes to prohibit PMZ before defining division by zero. In wheels and variants thereof [3], [2] PMZ as well as other fundamental algebraic properties are changed to support the definition of division by zero as non-real number.

In this work I propose to revisit the definition of PMZ (rather than attempt to define division by zero itself) by preserving information of the non-null part. I define $x \cdot 0$ as a non-real number I refer to as a null imaginary number $i_0(x)$. If $x \neq y$ then $i_0(x) \neq i_0(y)$. In spite of that, null imaginary numbers have the same real part which is zero, i.e. $\Re(i_0(x)) = \Re(i_0(xy)) = 0$. The theory of null imaginary numbers interprets that the value of x still does exist after

[1] please refer to [4] for ancient attempts to define division by zero.

multiplication by zero but in other unity or dimension. This implies that $y + \Re(i_0(x)) = y$ ($y \in \mathbb{R}_{\neq 0}$) but $y + i_0(x) = y + x \cdot 0 \neq y$. Based on this, I show $0/0 = 1$ does not cause logic inconsistencies to the elementary algebra. In the Section 2 I present notation, definitions and properties that build the algebraic field of the null imaginary numbers.

2 Notation, Definitions and Properties

In this section I present definitions and properties of the null imaginary numbers $\mathbb{I}_\emptyset$. I always denote both x and y as distinct non-zero real numbers unless differently stated. Similarly, **all usual properties of $\mathbb{R}$ does hold for $\mathbb{I}_\emptyset$ unless differently stated**.

2.1 Fundamentals

The process of multiplying $x \in \mathbb{R}_{\neq 0}$ by zero I refer to as *imagination* (Def. 2.1).

Definition 2.1 (Imagination) *I define a null imaginary number $i_0(x) \in \mathbb{I}_\emptyset, x \in \mathbb{R}_{\neq 0}$ as*

$$x \cdot 0 \quad = \quad i_0(x) \tag{1}$$

Imagination on x yields a *null imaginary number* (or just *n-imaginary*) $i_0(x)$ which is member of the null imaginary set of numbers $\mathbb{I}_\emptyset$ (Def. 2.2).

Definition 2.2 (The Set $\mathbb{I}_\emptyset$ of Null Imaginary Numbers)

$$\mathbb{I}_\emptyset \quad = \quad \{i_0(x) = x \cdot 0 | x \in \mathbb{R}_{\neq 0}\}. \tag{2}$$

No two real numbers leads to the same n-imaginary number on $\mathbb{I}_\emptyset$ (Def. 2.3). I extend this definition for the n-imaginary numbers too (Def. 2.4).

Definition 2.3 (Uniqueness of Imaginary Numbers)

$$x \neq y \quad \Leftrightarrow \quad i_0(x) \neq i_0(y),\ x, y \in \mathbb{R} \tag{3}$$

Definition 2.4 (Null imaginary Power) *Let $i_0(x) \in \mathbb{I}_\emptyset$, $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$. Then:*

$$x \cdot \prod_{i=1}^{n} 0 \quad = \quad i_0(x)^n \tag{4}$$

$$i_0(x) = i_0(x)^n \quad \Leftrightarrow \quad n = 1 \tag{5}$$

Despite the uniqueness on $\mathbb{I}_\emptyset$, all n-imarginary numbers preserve and share the same null real part (Def. 2.5).

Definition 2.5 (The Real Part $\Re(i_0(x))$ of a Null Imaginary Number $i_0(x)$)

$$\Re(i_0(x)) = 0, \forall x \in \mathbb{R} \tag{6}$$

In other words, Defs. 2.3 and 2.5 tell us that Eqs. 7 and 8 do hold, respectively.

$$x \cdot 0 \neq y \cdot 0 \tag{7}$$

$$\Re(x \cdot 0) = \Re(y \cdot 0) = 0 \tag{8}$$

The reverse process of imagination (i.e., dividing a null imaginary number by zero) I refer to as *realization* (Def. 2.6).

Definition 2.6 (Realization)

$$\frac{i_0(x)}{0} = x \tag{9}$$

2.2 Meaning of $0/0$ Based on $\mathbb{I}_\emptyset$

'Realization' (Def. 9) implies that $0/0 = 1$.

Property 1 *(The Set $\mathbb{I}_\emptyset$ defines $0/0 = 1$)*
Proof

$$\begin{aligned} \frac{i_0(x)}{0} &= x \text{ (Def. 9)} \\ 0 \cdot \frac{i_0(x)}{0} &= 0 \cdot x \\ \frac{0}{0} \cdot i_0(x) &= i_0(x) \text{ (Def. 2.1)} \end{aligned} \tag{10}$$

Since $i_0(x) = i_0(x)$, it follows that,

$$\frac{0}{0} = \frac{1}{1}$$

□

Since $x \cdot 0 \neq y \cdot 0$ (uniqueness, Def. 2.3), the result $0/0 = 1$ does not cause inconsistencies in mathematic. Besides, Eqs. 11, 12 do follow.

$$i_0(1) = 0 = \Re(i_0(1)) \tag{11}$$

$$\mathbb{I}_\emptyset \cap \mathbb{R} = \{0\} = \{i_0(1)\} \tag{12}$$

2.3 Real and Null Imaginary Division

Another form of *imagination* for a real number $x \in \mathbb{R}_{\neq 0}$ is division by zero, (Property 2).

Property 2 (*Imagination* by Division) *Let $x \in \mathbb{R}_{\neq 0}$. Then, $x/0$ is an n-imaginary number.*

Proof

$$\begin{aligned} \frac{x}{0} &= \frac{1}{0 \cdot x^{-1}} \\ \frac{x}{0} &= \frac{1}{i_0(x^{-1})} \end{aligned} \tag{13}$$

□

In fact, considering the property 3, (i.e., $y \cdot i_0(x) = i_0(xy)$), from Eq. 13 one gets the equality 14:

$$\begin{aligned} x \cdot i_0(x^{-1}) &= 0 \cdot 1 \\ i_0(x \cdot x^{-1}) &= i_0(1) \end{aligned} \tag{14}$$

Property 3 (Multiplication $\mathbb{R} \times \mathbb{I}_{\emptyset}$)

$$y \cdot i_0(x) = i_0(yx) \tag{15}$$

Proof

$$\begin{aligned} y \cdot i_0(x) &= y \cdot 0 \cdot x \\ &= y \cdot x \cdot 0 \\ &= yx \cdot 0 \\ &= i_0(yx) \end{aligned}$$

□

2.4 Elementary Algebra on $\mathbb{I}_{\emptyset}$

Property 4 (Sum) *Let $x, y \in \mathbb{R}$. Then*

$$i_0(x) + i_0(y) = i_0(x + y) \tag{16}$$

Proof:

$$\begin{aligned} i_0(x) + i_0(y) &= x \cdot 0 + y \cdot 0 \\ &= 0 \cdot (x + y) \\ &= i_0(x + y) \end{aligned}$$

□

Property 5 (Multiplication) *Let $i_0(x), i_0(y) \in \mathbb{I}_{\emptyset}$, $x, y \in \mathbb{R}$. Then*

$$i_0(x) \cdot i_0(y) = i_0(xy)^2 \tag{17}$$

Proof:

$$\begin{aligned}
i_0(x) \cdot i_0(y) &= 0 \cdot x \cdot 0 \cdot y \\
&= x \cdot y \cdot 0 \cdot 0 \\
&= xy \cdot 0 \cdot 0 \\
&= i_0(xy) \cdot 0, \textit{ (Def. 2.1)} \\
i_0(x) \cdot i_0(y) &= i_0(xy)^2, \textit{ (Def. 2.4)} \qquad (18)
\end{aligned}$$

□

Property 6 (Division) *Let* $i_0(x), i_0(y) \in \mathbb{I}_\emptyset$, $x, y \in \mathbb{R}_{\neq 0}$. *Then*

$$\frac{i_0(x)}{i_0(y)} = \frac{x}{y} \qquad (19)$$

Proof:

$$\begin{aligned}
\frac{i_0(x)}{i_0(y)} &= \frac{0}{0} \cdot \frac{x}{y} \\
\frac{i_0(x)}{i_0(y)} &= 1 \cdot \frac{x}{y}, \textit{ (Prop. 1)} \qquad (20)
\end{aligned}$$

□

Definition 2.7 (Multiplicative identity)

$$i_0(x) \cdot 1 = i_0(x) \qquad (21)$$

2.5 Null Subtraction

Let us consider the specific case $x - x$, $x \in \mathbb{R}_{\neq 0}$ in face of $\mathbb{I}_\emptyset$. Considering a pure real domain, Def. 2.5 ensures $\Re(x - x) = 0$. However, one concerning on the null-imaginary nature of $(x - x)$, may find out it can be neither $i_0(x)$ nor $i_0(-x)$. In fact, *realization* (Def. 2.6) tells us that $i_0(x)/0 = x \neq x - x$ as well as $i_0(-x)/0 = -x \neq x - x$. Thus, I define $x - x \in \mathbb{I}_\emptyset$ according to the Def. 2.8.

Definition 2.8 (Real Null Subtraction)

$$\begin{aligned}
x - x &= i_0(x - x) = i_0(\pm x) \qquad (22) \\
\Re(x - x) &= \Re(i_0(x - x)) = \Re(i_0(\pm x)) = 0 \qquad (23)
\end{aligned}$$

The subtraction of an n-imaginary number by itself is defined in Def. 7 considering Def. 2.8.

Property 7 (Null Imaginary Subtraction) *Let* $i_0(x) \in \mathbb{I}_\emptyset$*,* $x \in \mathbb{R}_{\neq 0}$*. Then*

$$i_0(x) - i_0(x) = i_0(\pm x)^2 \tag{24}$$

Proof:

$$\begin{aligned} i_0(x) - i_0(x) &= x \cdot 0 - x \cdot 0 && (25)\\ &= 0 \cdot (x - x) \\ &= 0 \cdot (i_0(\pm x)), \textit{ (Def. 2.8)} \\ &= i_0(\pm x)^2, \textit{ (Def. 2.4)} && (26) \end{aligned}$$

□

2.6 Euler's Identity and $i_0(\pm 1)$

The n-imaginary number $1 - 1 = i_0(\pm 1)$ seems to play a special rule for $\mathbb{I}_\emptyset$ because it lies in any subtraction of the type $x - x$. In fact, $x - x = x \cdot (1 - 1)$. Assuming the existence of $\mathbb{I}_\emptyset$ and admiting the usage of the unit imaginary $i = \sqrt{-1}$, $i_0(\pm 1)$ re-writes as the Euler's identity (Prop. 8).

Property 8 (Euler's Null Imaginary Identity) *Assuming the existence of* $\mathbb{I}_\emptyset$*, the Euler's identity becomes the (elementary) n-imaginary number* $i_0(\pm 1)$ *with null real-part, i.e.,*

$$\begin{aligned} e^{i\pi} + 1 &= i_0(\pm 1) && (27)\\ \Re(e^{i\pi} + 1) &= 0 && (28) \end{aligned}$$

Proof

$$\begin{aligned} e^{i\pi} + 1 &= 1 - 1 \\ e^{i\pi} + 1 &= i_0(\pm 1), \textit{ (Def. 2.8)} \\ \Re(e^{i\pi} + 1) &= \Re(i_0(\pm 1)) = 0, \textit{ (Def. 2.5)} \end{aligned}$$

□

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