

A Study on Set-Valuations of Signed Graphs

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Abstract: Let X be a non-empty ground set and $\mathcal{P}(X)$ be its power set. A set-labeling (or a set-valuation) of a graph G is an injective set-valued function $f : V(G) \rightarrow \mathcal{P}(X)$ such that the induced function $f^\oplus : E(G) \rightarrow \mathcal{P}(X)$ is defined by $f^\oplus(uv) = f(u) \oplus f(v)$, where $f(u) \oplus f(v)$ is the symmetric difference of the sets $f(u)$ and $f(v)$. A graph which admits a set-labeling is known to be a set-labeled graph. A set-labeling f of a graph G is said to be a set-indexer of G if the associated function f^\oplus is also injective. In this paper, we define the notion of set-valuations of signed graphs and discuss certain properties of signed graphs which admits certain types of set-valuations.

Key Words: Signed graphs, balanced signed graphs, clustering of signed graphs, set-labeled signed graphs.

AMS(2010): 05C78, 05C22.

§1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [4, 8, 13] and for the topics in signed graphs we refer to [14, 15]. Unless mentioned otherwise, all graphs considered here are simple, finite, undirected and have no isolated vertices.

1.1 An Overview of Set-Valued Graphs

Let X be a non-empty set and $\mathcal{P}(X)$ be its power set. A *set-labeling* (or a *set-valuation*) of a graph G is an injective function $f : V(G) \rightarrow \mathcal{P}(X)$ such that the induced function $f^\oplus : E(G) \rightarrow \mathcal{P}(X)$ is defined by $f^\oplus(uv) = f(u) \oplus f(v) \forall uv \in E(G)$, where \oplus is the symmetric difference of two sets. A graph G which admits a set-labeling is called an *set-labeled graph* (or a set-valued graph)(see [1]).

A *set-indexer* of a graph G is an injective function $f : V(G) \rightarrow \mathcal{P}(X)$ such that the induced function $f^\oplus : E(G) \rightarrow \mathcal{P}(X)$ is also injective. A graph G which admits a set-indexer is called a *set-indexed graph* (see [1]).

¹Received July 27, 2017, Accepted February 12, 2018.

Several types of set-valuations of graphs have been introduced in later studies and their properties and structural characteristics of such set-valued graphs have been done extensively.

1.2 Preliminaries on Signed Graphs

An edge of a graph G having only one end vertex is known as a *half edge* of G and an edge of G without end vertices is called *loose edge* of G .

A *signed graph* (see [14, 15]), denoted by $\Sigma(G, \sigma)$, is a graph $G(V, E)$ together with a function $\sigma : E(G) \rightarrow \{+, -\}$ that assigns a sign, either $+$ or $-$, to each ordinary edge in G . The function σ is called the *signature* or *sign function* of Σ , which is defined on all edges except half edges and is required to be positive on free loops.

An edge e of a signed graph Σ is said to be a *positive edge* if $\sigma(e) = +$ and an edge $\sigma(e)$ of a signed graph Σ is said to be a *negative edge* if $\sigma(e) = -$. The set E^+ denotes the set of all positive edges in Σ and the set E^- denotes the set of negative edges in Σ .

A simple cycle (or path) of a signed graph Σ is said to be *balanced* (see [3, 9]) if the product of signs of its edges is $+$. A signed graph Σ is said to be a *balanced signed graph* if it contains no half edges and all of its simple cycles are balanced. It is to be noted that the number of all negative signed graph is balanced if and only if it is bipartite.

Balance or imbalance is the basic and the most important property of a signed graph. The following theorem, popularly known as *Harary's Balance Theorem*, establishes a criteria for balance in a signed graph.

Theorem 1.1([9]) *The following statements about a signed graph are equivalent.*

- (i) *A signed graph Σ is balanced;*
- (ii) *Σ has no half edges and there is a partition (V_1, V_2) of $V(\Sigma)$ such that $E^- = E(V_1, V_2)$;*
- (iii) *Σ has no half edges and any two paths with the same end points have the same sign.*

Some balancing properties of certain types of signed graphs have been studied in [6, 7].

A signed graph Σ is said to be *clusterable* or *partitionable* (see [14, 15]) if its vertex set can be partitioned into subsets, called *clusters*, so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters. If $V(\Sigma)$ can be partitioned in to k subsets with the above mentioned conditions, then the signed graph Σ is said to be *k-clusterable*. In this paper, we study the 2-clusterability of signed graphs only.

Note that 2-clusterability always implies balance in a signed graph Σ . But, the converse need not be true. If all edges in Σ are positive edges, then Σ is balanced but not 2-clusterable.

In this paper, we introduce the notion of set-valuations of signed graphs and study the properties and characteristics of such signed graphs.

§2. Set-Labeled Signed Graphs

Motivated from the studies on set-valuations of signed digraphs in [2], and the studies on integer additive set-labeled signed graphs in [11], we define the notion of a set-labeling of a signed graph as follows.

Definition 2.1 Let X be a non-empty set and let Σ be a signed graph, with corresponding underlying graph G and the signature σ . An injective function $f : V(\Sigma) \rightarrow \mathcal{P}(X)$ is said to be a set-labeling (or set-valuation) of Σ if f is a set-labeling of the underlying graph G and the signature of Σ is defined by $\sigma(uv) = (-1)^{|f(u) \oplus f(v)|}$. A signed graph Σ together with a set-labeling f is known as a set-labeled signed graph (or set valued signed graph) and is denoted by Σ_f .

Definition 2.2 A set-labeling f of a signed graph Σ is said to be a set-indexer of Σ if f is a set-indexer of the underlying graph G .

If the context is clear, we can represent a set-valued signed graph or a set-indexed signed graph simply by Σ itself. In this section, we discuss the 2-clusterability and balance of set-valued signed graphs.

The following theorem establishes the existence of set-valuations for all signed graphs.

Theorem 2.3 Every signed graph admits a set-labeling (and a set-indexer).

Proof Let Σ be a signed graph whose vertex set is given by $V(\Sigma) = \{v_1, v_2, \dots, v_n\}$. Let $X = \{1, 2, 3, \dots, n\}$. Define a set-valued function $f : V(\Sigma) \rightarrow \mathcal{P}(X)$ such that $f(v_i) = \{i\}$, where $1 \leq i \leq n$. Clearly, f is an injective function. Then, $f^\oplus(v_i v_j) = \{i, j\}, \forall uv \in E(G)$. Note that f^\oplus is also an injective function and hence f is a set-indexer of Σ . \square

We say that two sets are of *same parity* if they are simultaneously even or simultaneously odd. If two sets are not of same parity, then they are said to be the sets of *opposite parity*. The signature of an edge of a set-valued signed graph can be determined in terms of the set-labels of its end vertices, as described in the following theorem.

Theorem 2.4 An edge e of a set-labeled signed graph is a positive edge if and only if the set-labels of its end vertices are of the same parity.

Proof Let f be a set-labeling of a given signed graph Σ . Assume that, an edge $e = v_i v_j$ be a positive edge in Σ . Then, $|f(v_i) \oplus f(v_j)| = |f(v_i) - f(v_j)| + |f(v_j) - f(v_i)|$ is an even number. That is, $|f(v_i) - f(v_j)|$ and $|f(v_j) - f(v_i)|$ are simultaneously even or simultaneously odd. Hence, we need to consider the following cases.

Case 1. Assume that both $|f(v_i) - f(v_j)|$ and $|f(v_j) - f(v_i)|$ are even. That is, both $|f(v_i) - f(v_i) \cap f(v_j)|$ and $|f(v_j) - f(v_i) \cap f(v_j)|$ are even. Then, we have

Subcase 1.1 Let $|f(v_i)|$ be an even integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)| = |f(v_i)| - |f(v_i) \cap f(v_j)|$, we have $|f(v_i) \cap f(v_j)|$ must also be even. Hence, as $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$ is even, we have $|f(v_j)|$ is even.

Subcase 1.2 Let $|f(v_i)|$ be an odd integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)| = |f(v_i)| - |f(v_i) \cap f(v_j)|$ is even, we have $|f(v_i) \cap f(v_j)|$ must be odd. Hence, as $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$ is even, we have $|f(v_j)|$ is odd.

Case 2. Assume that both $|f(v_i) - f(v_j)|$ and $|f(v_j) - f(v_i)|$ are odd. That is, both $|f(v_i) - f(v_i) \cap f(v_j)|$ and $|f(v_j) - f(v_i) \cap f(v_j)|$ are odd. Then, we have

Subcase 2.1 Let $|f(v_i)|$ be an even integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_i) \cap f(v_j)|$ must be odd. Hence, as $|f(v_j) - f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_j)|$ is even.

Subcase 2.2 Let $|f(v_i)|$ be an odd integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_i) \cap f(v_j)|$ must be even. Then, as $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_j)|$ is odd. \square

As a contrapositive of Theorem 2.4, we can prove the following theorem also.

Theorem 2.5 *An edge e of a set-labeled signed graph is a negative edge if and only if the set-labels of its end vertices are of the opposite parity.*

The following result is an immediate consequence of Theorems 2.4 and 2.5.

Corollary 2.6 *A set-valued signed graph Σ is balanced if and only if every cycle in Σ has even number of edges whose end vertices have opposite parity set-labels.*

Proof Note that the number of negative edges in any cycle of a balanced signed graph is even. Hence, the proof is immediate from Theorem 2.5. \square

The following theorem discusses a necessary and sufficient condition for a set-valued signed graph to be 2-clusterable.

Theorem 2.7 *A set-valued signed graph is 2-clusterable if and only if at least two adjacent vertices in Σ have opposite parity set-labels.*

Proof First, assume that at least two adjacent vertices in the set-valued signed graph Σ have opposite parity set-labels. If $e = v_i v_j$ be an edge of Σ such that $f(v_i)$ and $f(v_j)$ are of opposite parity, then $\sigma(v_i v_j) = -$. Then, we can find (U_1, U_2) be a partition of $V(\Sigma)$ such that U_1 contains one end vertex of every negative edge and U_2 contains the other end vertex of every negative edge. Therefore, Σ is 2-clusterable.

Conversely, assume that Σ is 2-clusterable. Then, there exist two non-empty subsets U_1 and U_2 of $V(\Sigma)$ such that $U_1 \cup U_2 = V(\Sigma)$. Since Σ is a connected signed graph, at least one vertex in U_1 is adjacent some vertices in U_2 and vice versa. Let $e = v_i v_j$ be such an edge in Σ . Since Σ is 2-clusterable, e is a negative edge and hence $f(v_i)$ and $f(v_j)$ are of opposite parity. This completes the proof. \square

Theorem 2.8 *Let f be a set-indexer defined on a signed graph Σ whose underlined graph G is an Eulerian graph. If Σ is balanced, then*

$$\sum_{e \in E(\Sigma)} |f^{\oplus}(e)| \equiv 0 \pmod{2}.$$

Proof Let the underlying graph G of Σ is Eulerian. Then, $G = \bigcup_{i=1}^k C_i$, where each C_i is a cycle such that C_i and C_j are edge-disjoint for $i \neq j$. Let E_i be the edge set of the cycle C_i .

Since f is a set-indexer of Σ , we have $f(E_i) \cap f(E_j) = \emptyset$, for $i \neq j$. Hence, we have

$$\sum_{e \in E(\S)} |f(e)| = \sum_{i=1}^k \sum_{e_i \in E_i} f(e_i). \quad (1)$$

Consider the cycle C_i . Let A_i be the set all positive edges and B_i be the set of all negative edges in the cycle C_i . Then, the set-labels of edges in A_i are of even parity and those of edges in B_i are of odd parity. That is, $|f^\oplus(e)| \equiv 0 \pmod{2}$ for all $e \in A_i$ and hence we have

$$\sum_{e \in A_i} |f(e)| \equiv 0 \pmod{2}. \quad (2)$$

Since Σ is balanced, the number of negative edges in C_i is even. Therefore, the number of elements in B_i must be even. That is, the number of edges having odd parity set-labels in C_i is even. Therefore, being a sum of even number of odd integers, we have

$$\sum_{e_i \in B_i} |f(e_i)| \equiv 0 \pmod{2}. \quad (3)$$

From Equations (2) and (3), we have

$$\sum_{e \in E_i} |f(e)| = \sum_{e_i \in A_i} |f(e_i)| + \sum_{e_i \in B_i} |f(e_i)| \equiv 0 \pmod{2}. \quad (4)$$

Therefore, by Equations (1) and (4), we can conclude that

$$\sum_{e \in E(\S)} |f^\oplus(e)| \equiv 0 \pmod{2}. \quad \square$$

From the above results, we infer the most important result on a set-valued signed graph as follows.

Theorem 2.9 *If a signed graph Σ admits a vertex set-labeling, then Σ is balanced.*

Proof Let Σ be a signed graph which admits a set-labeling. If all vertices of Σ have the same parity set-labels, then by Theorem 2.4, all edges of Σ are positive edges and hence Σ is balanced.

Next, let that Σ contains vertices with opposite parity set-labels. Let A_i be the set of all vertices with odd parity set-labels and B_i be the set of all even parity set-labels. First, assume v_i be a vertex in A_i whose adjacent vertices are in B_i . Then, v_i is one end vertex of some negative edges in Σ . If v_i is not in a cycle of Σ , then none of these negative edges will be a part in any cycle of Σ .

If v_i is an internal vertex of a cycle C , then it is adjacent to two vertices, say v_j and v_k , which are in B_i . Hence, the edges $v_i v_j$ and $v_i v_k$ are negative edges. If two vertices v_i and v_j are adjacent in the cycle C , then v_i is adjacent to one more vertex, say v_k and the vertex v_j is also adjacent to one more vertex v_l and in the cycle C , the edges $v_i v_k$ and $v_j v_l$ are negative

edges and the edge $v_i v_j$ is a positive edge. If the vertices v_i and v_j are not adjacent, then also each of them induce two negative, which may not be distinct always. However, in each case the number of negative edges will be even. This condition can be verified in all cases when any number element of A_i are the vertices of any cycle C in Σ . Hence, the number of negative edges in any cycle of a set-labeled signed graph is even. Hence, Σ is balanced.

Hence, in this case, the number negative edges in C will always be even. \square

It is interesting to check whether the converse of the above theorem is valid. In context of set-labeling of signed graphs, a necessary and sufficient condition for a signed graph Σ is to be balanced is given in the following theorem.

Theorem 2.10 *A signed graph Σ is balanced if and only if it admits a set-labeling.*

Proof The proof is an immediate consequence of Theorems 2.4 and 2.9. \square

In view of Theorems 2.4 and 2.10, we have

Theorem 2.11 *Any set-labeled signed graph is balanced.*

§3. Conclusion

In this paper, we have discussed the characteristics and properties of the signed graphs which admit set-labeling with a focus on 2-clusterability and balance of these signed graphs. There are several open problems in this area. Some of the open problems that seem to be promising for further investigations are following.

Problem 3.1 *Discuss the k -clusterability of different types of set-labeled signed graphs for $k > 2$.*

Problem 3.2 *Discuss the balance, 2-clusterability and general k -clusterability of other types of set-labeling of signed graphs such as topological set-labeling, topogenic set-labeling, graceful set-labeling, sequential set-labeling etc.*

Problem 3.3 *Discuss the balance and 2-clusterability and general k -clusterability of different set-labeling of signed graphs, with different set operations other than the symmetric difference of sets.*

Further studies on other characteristics of signed graphs corresponding to different set-labeled graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

Acknowledgement

The authors would like to dedicate this work to (Late) Prof. (Dr.) Belamannu Devadas Acharya, who had been the motivator and the role model for them and who have introduced the concept of set-valuations of graphs.

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