Cohen-Macaulay of Ideal \( I_2(G) \)

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Abstract: In this paper, we study the Cohen-Macaulay of ideal \( I_2(G) \), where \( I_2(G) = \langle xyz \mid x - y - z \text{ is a 2-path in } G \rangle \). Also, we determined the 2-projective dimension \( R \)-module, \( R/I_2(G) \) denoted by \( pd_2(G) \) of some graphs.

Key Words: Cohen-Macaulay, projective dimension, ideal, path.

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§1. Introduction

A simple graph is a pair \( G = (V, E) \), where \( V = V(G) \) and \( E = E(G) \) are the sets of vertices and edges of \( G \), respectively. A walk is an alternating sequence of vertices and connecting edges. A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A path with length \( n \) denotes by \( P_n \). In a graph \( G \), the distance between two distinct vertices \( x \) and \( y \), denoted by \( d(x, y) \), is the length of the shortest path connecting \( x \) and \( y \), if such a path exists: otherwise, we set \( d(x, y) = \infty \). The diameter of a graph \( G \) is \( \text{diam}(G) = \sup \{d(x, y) : x \text{ and } y \text{ are distinct vertices of } G\} \). Also, a cycle is a path that begins and ends on the same vertex. A cycle with length \( n \) denotes by \( C_n \). A graph \( G \) is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use \( K_n \) to denote the complete graph with \( n \) vertices. For a positive integer \( r \), a complete \( r \)-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph with part sizes \( m \) and \( n \) is denoted by \( K_{m,n} \). The graph \( K_{1,n-1} \) is called a star graph in which the vertex with degree \( n - 1 \) is called the center of the graph. For any graph \( G \), we denote \( N(x) = \{y \in V(G) : (x, y) \text{ is an edge of } G\} \). Recall that the projective dimension of an \( R \)-module \( M \), denoted by \( pd(M) \), is the length of the minimal free resolution of \( M \), that is, \( pd(M) = \max \{ I \mid \beta_{i,j}(M) \neq 0 \text{ for some } j \} \). There is a strong connection between the topology of the simplicial complex and the structure of the free resolution of \( K[\Delta] \). Let \( \beta_{i,j}(\Delta) \) denotes the \( N \)-graded Betti numbers of the Stanley-Reisner ring \( K[\Delta] \).

To any finite simple graph \( G \) with the vertex set \( V(G) = \{x_1, \ldots, x_n\} \) and the edge set \( E(G) \), one can attach an ideal in the Polynomial rings \( R = K[x_1, \ldots, x_n] \) over the field \( K \), where ideal \( l_2(G) \) is called the edge ideal of \( G \) such that \( l_2(G) = \langle xyz \mid x - y - z \text{ is a 2-path in } G \rangle \).

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path in $G >$. Also the edge ring of $G$, denoted by $K(G)$ is defined to be the quotient ring $K(G) = R/I_2(G)$. Edge ideals and edge rings were first introduced by Villarreal [9] and then they have been studied by many authors in order to examine their algebraic properties according to the combinatorial data of graphs. In this paper, we denote $S_n$ for a star graph with $n + 1$ vertices. Let $R = K[x_1, \cdots, x_n]$ be a polynomial ring over a field $K$ with the grading induced by $\deg(x_i) = d_i$, where $d_i$ is a positive integer. If $M = \bigoplus_{i=0}^{\infty} M_i$ is a finitely generated $R$-graded module over $R$, its Hilbert function and Hilbert series are defined by $H(M, i) = l(M_i)$ and $F(M, t) = \sum_{i=0}^{\infty} H(M, i)t^i$ respectively, where $l(M_i)$ denotes the length of $M_i$ as a $K$-module, in our case $l(M_i) = \dim_{K}(M_i)$.

§2. Cohen-Macaulay of Ideal $I_2(G)$ and $pd_2(G)$ of Some Graph $G$

**Definition 2.1** Let $G$ be a graph with vertex set $V$. Then a subset $A \subseteq V$ is a 2-vertex cover for $G$ if for every path $xyz$ of $G$ we have $\{x, y, z\} \cap A \neq \emptyset$. A 2-minimal vertex cover of $G$ is a subset $A$ of $V$ such that $A$ is a 2-vertex cover, and no proper subset of $A$ is a vertex cover for $G$. The smallest cardinality of a 2-vertex cover of $G$ is called the 2-vertex covering number of $G$ and is denoted by $\alpha_{02}(G)$.

**Example 2.2** Let $G$ be a graph shown in the figure. Then the set $\{x_2, x_4, x_7\}$ is a 2-minimal vertex cover of $G$ and $\alpha_{02}(G) = 3$.

![Figure 1](image_url)

**Definition 2.3** Let $G$ be a graph with vertex set $V$. A subset $A \subseteq V$ is a $k$-independent if for every $x$ of $A$ we have $\deg_{G[A]} \leq k - 1$. The maximum possible cardinality of an $k$-independent set of $G$, denoted $\beta_{0k}(G)$, is called the $k$-independence number of $G$. It is easy see that

$$\alpha_{02}(G) + \beta_{02}(G) = |V(G)|.$$ 

**Definition 2.4** Let $G$ be a graph without isolated vertices. Let $S = K[x_1, \cdots, x_n]$ the polynomial ring on the vertices of $G$ over some fixed field $K$. The 2-paths ideal $I_2(G)$ associated to the graph $G$ is the ideal of $S$ generated by the set of square-free monomials $x_i x_j x_r$ such that $\nu_i \nu_j \nu_r$.
Proposition 2.5 Let $\mathcal{S} = K[x_1, \cdots, x_n]$ be a polynomial ring over a field $K$ and $G$ a graph with vertices $\nu_1, \cdots, \nu_n$. If $P$ is an ideal of $R$ generated by $A = \{x_{i_1}, \cdots, x_{i_k}\}$ then $P$ is a minimal prime of $I_2(G)$ if and only if $A$ is a 2-minimal vertex cover of $G$.

Proof It is easy see that $I_2(G) \subseteq P$ if and only if $A$ is a 2-vertex cover of $G$. Now, let $A$ is a 2-minimal vertex cover of $G$. By Proposition 5.1.3 [9] any minimal prime ideal of $I_2(G)$ is a face ideal thus $P$ is a minimal prime of $I_2(G)$. The converse is clear. □

Corollary 2.6 If $G$ is a graph and $I_2(G)$ its 2-path ideal, then

$$ht(I_2(G)) = \alpha_{02}(G).$$

Proof It follows from Proposition 5 and the definition of $\alpha_{02}(G)$. □

Definition 2.7 A graph $G$ is 2-unmixed if all of its 2-minimal vertex covers have the same cardinality.

Definition 2.8 A graph $G$ with vertex set $V(G) = \{\nu_1, \nu_2, \cdots, \nu_n\}$ is 2- Cohen-Macaulay over field $K$ if the quotient ring $K[x_1, \cdots, x_n]/I_2(G)$ is Cohen-Macaulay.

Proposition 2.9 If $G$ is a 2-Cohen-Macaulay graph, then $G$ is 2-unmixed.

Proof By Corollary 1.3.6 [9], $I_2(G) = \bigcap_{P \in \text{min}(I_2(G))} P$. Since $R/I_2(G)$ is Cohen-Macaulay, all minimal prime ideals of $I_2(G)$ have the same height. Then, by Proposition 5, all 2-minimal vertex cover of $G$ have the same cardinality, as desired. □

Proposition 2.10 If $G$ is a graph and $G_1, \cdots, G_s$ are its connected components, then $G$ is 2-Cohen-Macaulay if and only if for all $i$, $G_i$ is Cohen-Macaulay.

Proof Let $R = K[V(G)]$ and $R_i = K[V(G_i)]$ for all $i$. Since

$$R/I_2(G) \cong R_1/I_2(G_1) \otimes_K \cdots \otimes_K R_s/I_2(G_s).$$

Hence the results follow from Corollary 2.2.22 [9]. □

Definition 2.11 For any graph $G$ one associates the complementary simplicial complex $\Delta_2(G)$, which is defined as

$$\Delta_2(G) := \{A \subseteq V | A \text{ is 2-independent set in } G\}.$$

This means that the facets of $\Delta_2(G)$ are precisely the maximal 2-independent sets in $G$, that is the complements in $V$ of the minimal 2-vertex covers. Thus $\Delta_2(G)$ precisely the Stanley-Reisner complex of $I_2(G)$.
It is easy see that \( \omega(\Delta_2(G)) = \{(x, y, z) \mid xyz \in P_3(G)\} \). Therefore \( I_2(G) = I_{\Delta_2(G)} \), and so \( G \) is \( 2 - C - M \) graph if and only if the simplicial complex \( \Delta_2(G) \) is cohen-Macaulay.

Now, we can show the following propositiori.

**Proposition 2.12** The following statements hold:

(a) For any \( n \geq 1 \) the complete graph \( K_n \) is cohen-Macaulay;

(b) The complete bipartite graph \( K_{m,n} \) is cohen-Macaulay if and only if \( m + n \leq 4 \).

**Proof** (a) Since \( \Delta_2(K_n) = \langle \{x, y\} \mid x, y \in V(K_n) \rangle \), thus \( \Delta_2(K_n) \) is connected \( l \)-dimensional simplicial complex, then by Cororary 5.3.7 [9], \( \Delta_2(K_n) \) is cohen-Macaulay so \( K_n \) is cohen-Macaulay.

(b) If \( m + n \leq 4 \), then \( K_{m,n} \cong P_2, P_3, C_4 \). It is easy to see that \( \Delta_2(K_{m,n}) \) is c. So \( K_{m,n} \) is cohen-Macaulay.

Conversely, let \( K_{m,n} \) is cohen-Macaulay and \( m + n \geq 5 \). Take \( V_1 = \{x_1, \cdots, x_n\} \) and \( V_2 = \{y_1, \cdots, y_m\} \) are the partite sets of \( K_{m,n} \). One has

\[
\Delta_2(K_{m,n}) = \langle \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \leq i \leq n, 1 \leq j \leq m \rangle.
\]

Since \( m + n \geq 5 \), \( \Delta_2(K_{m,n}) \) is not pure simplicial complex. Then, by 5.3.12 [9] \( \Delta_2(K_{m,n}) \) is not cohen-Macaulay, a contradiction, as desired.

Now, we present a result about the Hilbert series of \( K[\Delta_2(K_n)] \) and \( K[\Delta_2(K_{m,n})] \).

**Proposition 2.13** If \( \Delta_2(K_n) \) and \( \Delta_2(K_{m,n}) \) are the complemenary simplicial complexes \( K_n \) and \( K_{m,n} \) respectively, then

(a) \( F(K[\Delta_2(K_n)], z) = 1 + nz/(1 - z) + n(n - 1)/2(1 - z)^2 \);

(b) \( F(K[\Delta_2(K_{m,n})], z) = 1/(1 - z)^n + 1/(1 - z)^m + mnz^2/(1 - z)^2 - 1 \).

**Proof** (a) Since \( \Delta_2(K_n) = \langle \{x, y\} \mid x, y \in V(K_n) \rangle \) hence dime \( \Delta_2(K_n) = 1 \) and \( f_{-1}(K_n) = 1 \), \( f_0(K_n) = n \) and \( f_1(K_n) = \binom{n}{2} = n(n - 1)/2 \). By Corollary 5.4.5 [9]. We have

\[
F(K[\Delta_2(K_n)], z) = 1 + nz/1 - z + n(n - 1)/2. z^2/2(1 - z)^2.
\]

(b) Let \( \{x_1, \cdots, x_n\} \) and \( \{y_1, \cdots, y_m\} \) are the parties sets of \( K_{m,n} \). Since

\[
\Delta_2(K_{m,n}) = \langle \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \leq i \leq n, 1 \leq j \leq m \rangle
\]

Then it is easy see that \( f_1(\Delta_2(K_{m,n})) = f_1(\Delta(K_{m,n}))+ mn \) and \( f_i(\Delta_2(K_{m,n})) = f_i(\Delta(K_{m,n})) \) for all \( i \neq 1 \). In the other hand, by 6.6.6[9], \( F(K[\Delta_2(K_n)], z) = 1/(1 - z)^n - 1 \). Thus

\[
F(K[\Delta_2(K_{m,n})], z) = 1/(1 - z)^n + 1/(1 - z)^m + mnz^2/(1 - z)^2 - 1.
\]

This completes the proof.

**Corollary 2.14** \( F(K[\Delta_2(S_n)], z) = 1/(1 - z)^n + nz^2/(1 - z)^2 + z/(1 - z) \).
Proof It follows from Proposition 2.13 with assume $m = 1$. \qed

In this section we mainly present basic properties of 2-shellable graphs.

Lemma 2.15 Let $G$ be a graph and $x$ be a vertex of degree 1 in $G$ and let $y \in N(x)$ and $G' = G - (\{y\} \cup N(y))$. Then $\Delta_2(G') = lK_{\Delta_2(G)}(\{x, y\})$. Moreover $F$ is a facet of $\Delta_2(G')$ if and only if $F \cup \{x, y\}$ is a facet of $\Delta_2(G)$.

Proof (a) Let $F \in lK_{\Delta_2(G)}(\{x, y\})$. Then $F \in \Delta_2(G)$, $x, y \notin F$ and $F \cup \{x, y\} \in \Delta_2(G)$. This implies that $(F \cup \{x, y\}) \cap N[y] = \emptyset$ and $F \subseteq (V - \{x, y\}) \cup N[y] = (V - y) \cup N[y] = V(G')$. Thus $F$ is 2-independent in $G'$, it follows that $F \in \Delta_2(G')$. Conversely let $F \in \Delta_2(G')$, then $F$ is 2-independent in $G'$ and $F \cap (x \cup [y]) = \emptyset$. Therefore $F \cup \{x, y\}$ is 2-independent in $G$ and so $F \cup \{x, y\} \in \Delta_2(G)$, $F \cup \{x, y\} = \emptyset$. Thus $F \in lK_{\Delta_2(G)}(\{x, y\})$. Finally from part one follows that $F$ is a facet of $\Delta_2(G')$ if and only if $F \cup \{x, y\}$ is a facet of $\Delta_2(G)$. \qed

Definition 2.16 Fix a field $K$ and set $R = K[x_1, \ldots, x_n]$. If $G$ is a graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$, we define the projective dimension of $G$ to be the 2-projective dimension $R$-module $R/I_2(G)$, and we will write $pd_2(G) = pd(R/I_2(G))$.

Proposition 2.17 If $G$ is a graph and $\{x, y\}$ is an edge of $G$, then

$$P_2(G) \leq \max \{P_2(G - (N[x] \cup N[y])) + \deg(x) + \deg(y), |N[x] \cap N[y]|, P_2(G - x) + 1, P_2(G - y) + 1\}.$$

Proof Let $N[x] = \{x_1, \ldots, x_ξ\}$ and $N[y] = \{y_1, \ldots, y_ξ\}$. It is easy to see that

$$I_2(G) : xy = (I_2(G) - (N[x] \cup N[y]), x_1, \ldots, x_ξ, y_1, \ldots, y_ξ).$$

Now, let

$$R' = K \left[ V \left( G - (N[x] \cup N[y]) \right) \right].$$

Then

$$\text{depth}(R/I_2(G) : xy) = \text{depth}(R'/I_2(G - (N[x] \cup N[y])).$$

And so by Auslander-Buchsbaum formula, we have

$$pd_2(R/I_2(x) : xy) = pd_2(G - (N[x] \cup N[y])) + \deg(x) + \deg(y) - |N[x] \cap N[y]|,$$

$$pd_2(R/I_2(x), x) = pd_2(G - x) + 1,$$

$$pd_2(R/I_2(x), y) = pd_2(G - y) + 1.$$
it follows that

\[ P_2(G) \leq \max \{ P_2(G - (N[x] \cup N[y])) + \deg(x) + \deg(y) \]

\[ - |N[x] \cap N[y]|, P_2(G - x) + 1, P_2(G - y) + 1 \}. \]

\[ \Box \]

**Proposition 2.18** Let \( G \) be a graph and \( I_2(G) \) is path ideal of \( G \). Then

\[ \text{Bight}(I_2(G)) \leq \text{pd}_2(G). \]

**Proof** Let \( P \) be a minimal vertex cover with maximal cardinality. Then by Proposition 2.5, \( P \) is an associated prime of \( R/I_2(G) \), so

\[ \text{pd}_2(G) = \text{pd}(R/I_2(G)) \geq \text{pd}_{R_p}(R_p/I_2(G)R_p) = \dim R_p = \text{ht}P. \]

\[ \Box \]

**Proposition 2.19** Let \( K_n \) denote the complete graph on \( n \) vertices and let \( K_{m,n} \) denote the complete bipartite graph on \( m + n \) vertices.

(a) \( \text{pd}_2(K_n) = n - 2 \);

(b) \( \text{pd}_2(K_{m,n}) = m + n - 2 \).

**Proof** (a) The proof is by induction on \( n \). If \( n = 2 \) or \( 3 \), then the result easy follows. Let \( n \geq 4 \) and suppose that for every complete graphs \( K_n \) of other less than \( n \) the result is true. Since \( \text{Bight}(I_2(K_n)) = n - 2 \) then by Proposition \( \text{pd}_2(K_n) \geq n - 2 \). On the other hand by the inductive hypothesis, we have \( \text{pd}_2(K_{n-1}) = n - 3 \). So by Proposition 2.17,

\[ \text{pd}_2(K_n) \leq \max \{ n - 2, n - 2 \}. \]

(b) Again we use by induction on \( m + n \). If \( m + n = 2 \) or \( 3 \), then it is easy to see that \( \text{pd}_2(K_{m,n}) = 0 \) or \( 1 \). Let \( m + n \geq 4 \) and suppose that for every complete bipartite graph \( K_{m,n} \) of order less than \( m + n \) the result is true. Since \( \text{Bight}(I_2(K_{m,n})) = m + n - 2 \) then \( \text{pd}_2(K_{m,n}) \geq m + n - 2 \). Also, by the inductive hypothesis we have \( \text{pd}_2(K_{m-1,n}) = m + n - 3 \) and \( \text{pd}_2(K_{m,n-1}) = m + n - 3 \). So by Proposition 2.17,

\[ \text{pd}_2(K_{m,n}) \leq \max \{ m + n - 2, \text{pd}_2(K_{m-1,n}) + 1, \text{pd}_2(K_{m,n-1}) + 1 = m + n - 2 \}. \]

This completes the proof. \[ \Box \]

**Corollary 2.20** Let \( S_n \) denote the star graph on \( n + 1 \) vertices and \( S_{m,n} \) denote the double star, then \( \text{pd}_2(S_{m,n}) = m + n \).

**Proof** It follows from Proposition 2.19 with assume \( m = 1 \) and it is easy to see that \( \text{Bight}I_2(S_{m,n}) = m + n \), and so by Proposition 2.17, it follows that

\[ \text{pd}_2(S_{m,n}) = m + n. \]

\[ \Box \]
References


