Equitable Coloring on Triple Star Graph Families

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Abstract: An equitable k-coloring of a graph G is a proper k-coloring of G such that the sizes of any two color class differ by at most one. In this paper we investigate the equitable chromatic number for the Central graph, Middle graph, Total graph and Line graph of Triple star graph K_{1,n,n,n} denoted by C(K_{1,n,n,n}), M(K_{1,n,n,n}), T(K_{1,n,n,n}) and L(K_{1,n,n,n}) respectively.

Key Words: Equitable coloring, Smarandachely equitable k-coloring, triple star graph, central graph, middle graph, total graph and line graph.

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§2. Preliminaries

For a given graph \( G = (V, E) \) we do an operation on \( G \), by subdividing each edge exactly once and joining all the non-adjacent vertices of \( G \). The graph obtained by this process is called central graph of \( G \) \([1]\) and is denoted by \( C(G) \).

The line graph \([6]\) of a graph \( G \), denoted by \( L(G) \) is a graph whose vertices are the edges of \( G \) and if \( u, v \in E(G) \) then \( uv \in E(L(G)) \) if \( u \) and \( v \) share a vertex in \( G \).

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). The middle graph \([2]\) of \( G \) denoted by \( M(G) \) is defined as follows. The vertex set of \( M(G) \) is \( V(G) \cup E(G) \) in which two vertices \( x, y \) are adjacent in \( M(G) \) if the following condition hold:

1. \( x, y \in E(G) \) and \( x, y \) are adjacent in \( G \);
2. \( x \in V(G), \ y \in E(G) \) and they are incident in \( G \).

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). The total graph \([1,2]\) of \( G \) is denoted by \( T(G) \) and is defined as follows. The vertex set of \( T(G) \) is \( V(G) \cup E(G) \). Two vertices \( x, y \) in the vertex set of \( T(G) \) is adjacent in \( T(G) \), if one of the following holds:

1. \( x, y \) are in \( V(G) \) and \( x \) is adjacent to \( y \) in \( G \);
2. \( x, y \) are in \( E(G) \) and \( x, y \) are adjacent in \( G \);
3. \( x \) is in \( V(G) \), \( y \) is in \( E(G) \) and \( x, y \) are adjacent in \( G \).

Triple star \( K_{1,n,n,n} \) \([9]\) is a tree obtained from the double star \([2]\) \( K_{1,n,n} \) by adding a new pendant edge of the existing \( n \) pendant vertices. It has \( 3n + 1 \) vertices and \( 3n \) edges.

§3. Equitable Coloring on Central Graph of Triple Star Graph

Algorithm 1.

Input: The number ‘\( n \)’ of \( k_{1,n,n,n} \).
Output: Assigning equitable colouring for the vertices in \( C(K_{1,n,n,n}) \).

begin
for \( i = 1 \) to \( n \)
{ 
  \( V_1 = \{ e_i \} \)
  \( C(e_i) = i; \)
  \( V_2 = \{ a_i \} \)
  \( C(a_i) = i; \)
}
\( V_3 = \{ v \}; \)
\( C(v) = n + 1; \)
for $i = 2$ to $n$
{
$V_4 = \{v_i\}$;
$C(v_i) = i - 1$;
$V_5 = \{w_i\}$;
$C(w_i) = i - 1$;
$V_6 = \{u_i\}$;
$C(u_i) = i - 1$;
}
$C(v_1) = n$;
$C(w_1) = n$;
$C(u_1) = n$;
for $i = 1$ to $5$
{
$V_7 = \{s_i\}$;
$C(s_i) = n + 1$;
}
for $i = 6$ to $n$
{
$V_8 = \{s_i\}$;
$C(s_i) = i$;
}
$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8$;
end

**Theorem 3.1** For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number

$$\chi = [C(K_{1,n,n,n})] = n + 1.$$  

*Proof* Let $\{v_i : 1 \leq i \leq n\}$, $\{w_i : 1 \leq i \leq n\}$ and $\{u_i : 1 \leq i \leq n\}$ be the vertices in $K_{1,n,n,n}$. The vertex $v$ is adjacent to the vertices $v_i(1 \leq i \leq n)$. The vertices $v_i(1 \leq i \leq n)$ is adjacent to the vertices $w_i(1 \leq i \leq n)$ and the vertices $w_i(1 \leq i \leq n)$ is adjacent to the vertices $u_i(1 \leq i \leq n)$.

By the definition of central graph on $K_{1,n,n,n}$, let the edges $vv_i, v_iw_i$ and $w_iu_i (1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i, a_i, s_i(1 \leq i \leq n)$ respectively.
Clearly,
\[ V[C(K_{1,n,n,n})] = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \]
\[ \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \]
\[ \bigcup \{a_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \]

The vertices \(v\) and \(u_i (1 \leq i \leq n)\) induces a clique of order \(n + 1\) (say \(k_{n+1}\)) in \(C[K_{1,n,n,n}]\). Therefore
\[ \chi = [C(K_{1,n,n,n})] \geq n + 1 \]

Now consider the vertex set \(V[C(K_{1,n,n,n})]\) and the color class \(C = \{c_1, c_2, c_3, \cdots c_{n+1}\}\). Assign an equitable coloring to \(C(K_{1,n,n,n})\) by Algorithm 1. Therefore
\[ \chi = [C(K_{1,n,n,n})] \leq n + 1. \]

An easy check shows that \(||v_i - v_j|| \leq 1\). Hence
\[ \chi = [C(K_{1,n,n,n})] = n + 1. \]

\section*{§4. Equitable Coloring on Line graph of Triple Star Graph}

\textbf{Algorithm 2.}

\textbf{Input:} The number ‘n’ of \(K_{1,n,n,n}\);  
\textbf{Output:} Assigning equitable coloring for the vertices in \(L(K_{1,n,n,n})\).

\begin{verbatim}
begin
    for \(i = 1\) to \(n\)  
        \{  
            \(V_1 = \{e_i\}\);  
            \(C(e_i) = i\);  
            \(V_2 = \{s_i\}\);  
            \(C(s_i) = i\);  
        \}
    for \(i = 2\) to \(n\)  
        \{  
            \(V_3 = \{a_i\}\);  
            \(C(a_i) = i - 1\);  
        \}
\end{verbatim}
\[ C(a_1) = n; \]
\[ V = V_1 \cup V_2 \cup V_3; \]
end

**Theorem 4.1** For any triple star graph \( K_{1,n,n,n} \) the equitable chromatic number,

\[ \chi = [L(K_{1,n,n,n})] = n. \]

**Proof** Let \( \{v_i : 1 \leq i \leq n\}, \{w_i : 1 \leq i \leq n\} \) and \( \{u_i : 1 \leq i \leq n\} \) be the vertices in \( K_{1,n,n,n} \). The vertex \( v \) is adjacent to the vertices \( v_i(1 \leq i \leq n) \) with edges \( e_i(1 \leq i \leq n) \). The vertices \( w_i(1 \leq i \leq n) \) is adjacent to the vertices \( w_i(1 \leq i \leq n) \) with edges \( a_i(1 \leq i \leq n) \). The vertices \( u_i(1 \leq i \leq n) \) is adjacent to the vertices \( u_i(1 \leq i \leq n) \) with edges \( s_i(1 \leq i \leq n) \).

By the definition of line graph on \( K_{1,n,n,n} \) the edges \( e_i, a_i, s_i(1 \leq i \leq n) \) of \( K_{1,n,n,n} \) are the vertices of \( L(K_{1,n,n,n}) \). Clearly

\[ V[L(K_{1,n,n,n})] = \{e_i : 1 \leq i \leq n\} \cup \{a_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \]

The vertices \( e_i(1 \leq i \leq n) \) induces a clique of order \( n \) (say \( K_n \)) in \( L(K_{1,n,n,n}) \). Therefore

\[ \chi = [L(K_{1,n,n,n})] \geq n. \]

Now consider the vertex set \( V[L(K_{1,n,n,n})] \) and the color class \( C = \{c_1, c_2, \cdots, c_n\} \).

Assign an equitable coloring to \( L(K_{1,n,n,n}) \) by Algorithm 2. Therefore

\[ \chi = [L(K_{1,n,n,n})] \leq n. \]

An easy check shows that \( ||v_i| - |v_j|| \leq 1 \). Hence

\[ \chi = [L(K_{1,n,n,n})] = n. \]

§5. Equitable Coloring on Middle and Total Graphs of Triple Star Graph

**Algorithm 3.**

**Input:** The number ‘\( n \)’ of \( K_{1,n,n,n} \);

**Output:** Assigning equitable coloring for the vertices in \( M(K_{1,n,n,n}) \) and \( T(K_{1,n,n,n}) \).
begin
for $i = 1$ to $n$
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{s_i\}$;
$C(s_i) = i$;
}
$V_3 = \{v\}$;
$C(v) = n + 1$;
for $i = 2$ to $n$
{
$V_4 = \{v_i\}$;
$C(v_i) = i - 1$;
}
$C(v_1) = n$;
for $i = 3$ to $n$
{
$V_5 = \{a_i\}$;
$C(a_i) = i - 2$;
}
$C(a_1) = n + 1$;
$C(a_2) = n + 1$;
for $i = 4$ to $n$
{
$V_6 = \{w_i\}$
$C(w_i) = i - 3$;
}
$C(w_1) = n - 1$;
$C(w_2) = n$;
\( C(w_3) = n + 1; \)
for \( i = 1 \) to \( n \)
\{ 
\[ V_7 = \{ u_i \}; \]
\[ C(u_i) = i + 1; \]
\}
\[ V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \]
end

**Theorem 5.1** For any triple star graph \( K_{1,n,n,n} \) the equitable chromatic number, 
\[ \chi = [M(K_{1,n,n,n})] = n + 1, \ n \geq 4. \]

**Proof** Let \( V(K_{1,n,n,n}) = \{ v \} \cup \{ v_i : 1 \leq i \leq n \} \cup \{ w_i : 1 \leq i \leq n \} \cup \{ u_i : 1 \leq i \leq n \} \).

By the definition of middle graph on \( K_{1,n,n,n} \) each edge \( vv_i, v_iw_i \) and \( w_iu_i (1 \leq i \leq n) \) in \( K_{1,n,n,n} \) are subdivided by the vertices \( e_i, w_i, s_i (1 \leq i \leq n) \) respectively. Clearly
\[
V[M(K_{1,n,n,n})] = \{ v \} \bigcup \{ v_i : 1 \leq i \leq n \} \bigcup \{ w_i : 1 \leq i \leq n \} \\
\bigcup \{ u_i : 1 \leq i \leq n \} \bigcup \{ e_i : 1 \leq i \leq n \} \\
\bigcup \{ a_i : 1 \leq i \leq n \} \bigcup \{ s_i : 1 \leq i \leq n \}
\]

The vertices \( v \) and \( e_i (1 \leq i \leq n) \) induces a clique of order \( n + 1 \) (say \( k_{n+1} \)) in \( [M(K_{1,n,n,n})] \).

Therefore
\[ \chi = [M(K_{1,n,n,n})] \geq n + 1. \]

Now consider the vertex set \( V[M(K_{1,n,n,n})] \) and the color class \( C = \{ c_1, c_2, \cdots , c_{n+1} \} \).

Assign an equitable coloring to \( M(K_{1,n,n,n}) \) by Algorithm 3. Therefore
\[ \chi = M[K_{1,n,n,n}] \leq n + 1, \ ||v_i| - |v_j|| \leq 1. \]

Hence
\[ \chi = [M(K_{1,n,n,n})] = n + 1 \ \forall n \geq 4. \]

**Theorem 5.2** For any triple star graph \( K_{1,n,n,n} \) the equitable chromatic number,
\[ \chi = [T(K_{1,n,n,n})] = n + 1, \ n \geq 4. \]

**Proof** Let \( V(K_{1,n,n,n}) = \{ v \} \cup \{ v_i : 1 \leq i \leq n \} \cup \{ w_i : 1 \leq i \leq n \} \cup \{ u_i : 1 \leq i \leq n \} \) and \( E(K_{1,n,n,n}) = \{ e_i : 1 \leq i \leq n \} \cup \{ a_i : 1 \leq i \leq n \} \cup \{ s_i : 1 \leq i \leq n \}. \)
By the definition of Total graph, the edge $vv_i, v_iw_i$ and $w_iu_i (1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i, a_i$ and $s_i (1 \leq i \leq n)$ respectively. Clearly

$$V[T(K_{1,n,n,n})] = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\}.$$ 

The vertices $v$ and $e_i (1 \leq i \leq n)$ induces a clique of order $n+1$ (say $k_{n+1}$) in $T(K_{1,n,n,n})$. Therefore

$$\chi[T(K_{1,n,n,n})] \geq n+1, \quad n \geq 4.$$ 

Now consider the vertex set $V(T(K_{1,n,n,n}))$ and the color class $C = \{c_1, c_2, \ldots, c_{n+1}\}$. Assign an equitable coloring to $T(K_{1,n,n,n})$ by Algorithm 3. Therefore

$$\chi[T(K_{1,n,n,n})] \leq n+1, \quad n \geq 4, \quad ||v_i|| - ||v_j|| \leq 1.$$ 

Hence

$$\chi[T(K_{1,n,n,n})] = n + 1, \forall n \geq 4. \quad \square$$

References


