International J.Math. Combin. Vol.2(2018), 129-133

Graph Operations on Zero-Divisor Graph of Posets

N.Hosseinzadeh

(Department of Mathematics, Islamic Azad University, Dezful Branch, Dezful, Iran)

E-mail: narges.hosseinzadeh@gmail.com

Abstract: We know that some large graphs can be constructed from some smaller graphs by using graphs operations. Many properties of such large graphs are closely related to those of the corresponding smaller ones. In this paper we investigate some operations of zero-divisor graph of posets.

Key Words: Poset, zero-divisor graph, graph operation.

AMS(2010): 06A11,05C25.

§1. Introduction

In [2], Beck, for the first time, studied zero-divisor graphs of the commutative rings. Later, D. F. Anderson and Livingston investigated nonzero zero-divisor graphs of the rings (see [1]). Some researchers also studied the zero-divisor graph of the commutative rings. Subsequently, others extended the study to the commutative semigroups with zero. These can be seen in [3, 5, 7, 8].

Assume $(P, \leq)$ is a poset (i.e., P is a partially ordered set) with the least element 0. For every $x, y \in P$, defined of $L(x, y) = \{z \in P | z \leq x \text{ and } z \leq y\}$. x is a zero-divisor element of P if $l(x, y) = 0$, for some $0 \neq y \in P$. $\Gamma(P)$ is the zero-divisor graph of poset P, where the its vertex set consists of nonzero zero-divisors elements of P and x is adjacent to y if only if $L(x, y) = \{0\}$.

In this paper, P denotes a poset with the least element 0 and $Z(P)$ is nonzero zero-divisor elements of P. The zero-divisor graph is undirected graph with vertices $Z(S)$ such that for every distinct $x, y \in Z(S)$, x and y are adjacent if only if $L(x, y) = \{0\}$. Throughout this paper, G always denotes a zero-divisor graph which is a simple graph (i.e., undirected graph without loops and multiple) and the set vertices of G show $V(G)$ and the set edges of G denotes $E(G)$. The degree of vertex x is the number of edges of G intersecting x. $N(x)$, which is the set of vertices adjacent to vertex x, is called the neighborhood of vertex x. If n is a (finite or infinite) natural number, then an n-partite graph is a graph, which is a set of vertices that can be partitioned into subsets, each of which edges connects vertices of two different sets. A complete n- partite graph is a n- partite graph such that every vertex is adjacent to the vertices which are in a different part. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is called an induced subgraph of G if for every $x, y \in V(H)$, $\{x, y\} \in E(G)$. A subgraph H of G is called a clique if H is a complete graph. The clique number $\omega(G)$ of G is the least upper

[1]Received November 17, 2017, Accepted May 30, 2018.

bound of the cliques sizes of G.

Many large graphs can be constructed by expanding small graphs, thus it is important to know which properties of small graphs can be transfered to the expanded ones, for example Wang in [6] proved that the lexicographof vertex transitive graphs is also vertex transitive as well as the lexicographic product of edge transitive graphs. Specapan in [9] found the fewest number of vertices for Cartesian product of two graphs whose removal from the graph results in a disconnected or trivial graph. Motivated by these, we consider five kinds of graph products as the expander graphs which is described below and we can verify if regard the product of them can be regarded as a Cayley graph of the semigroup which is made by their product underlying semigroup and if the answer is positive does it inherit Col-Aut-vertex property of from the precedents. Let $\Gamma = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of G. An edge joins the vertex u to the vertex v is denoted by (u, v).

In [10], the authors described the following definition:

Definition 1.1 *A simple graph G is called a compact graph if G does not contain isolated vertices and for each pair x and y of non-adjacent vertices of G, there exists a vertex z with $N(x) \cup N(y) \subseteq N(z)$.*

Definition 1.2 *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. $\Gamma = (V, E)$, the product of them is a graph with vertex set $V = V_1 \times V_2$, and two vertices (u_1, u_2) is adjacent to (v_1, v_2) in Γ if one of the relevant conditions happen depending on the product.*

(1) Cartesian product. *u_1 is adjacent to v_1 in Γ_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2;*

(2) Tensor product. *u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2;*

(3) Strong product. *u_1 is adjacent to v_1 in Γ_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2 or u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2;*

(4) Lexicographic. *u_1 is adjacent to v_1 in Γ_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2;*

(5) Co-normal product. *u_1 is adjacent to v_1 in Γ_1 or u_2 is adjacent to v_2 in Γ_2;*

(6) Modular product. *u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2 or u_1 is not adjacent to v_1 in Γ_1 and u_2 is not also adjacent to v_2 in Γ_2.*

§2. Preliminary Notes

In this section, we recall some lemmas and definitions. from Dancheny Lu and Tongsue We in [10], the authors described following definition.

Definition 2.1 *A simple graph G is called a compact graph if G does not contain isolated vertices and for each pair x and y of non-adjacent vertices of G, there exists a vertex z with $N(x) \cup N(y) \subseteq N(z)$.*

It has been showed the following theorem in [10].

Theorem 2.2 *A simple graph G is the zero-divisor graph of a poset if and only if G is a* *compact graph.*

§3. Cartesian Product

Through this section, we assume that P and Q are two posets with the least element 0. Assume G and H are in zero-divisors graphs of P and Q, respectively. $N(x)$ and $N(a)$ are neighborhoods in G and H , respectively, where $x \in V(P)$ and $a \in V(Q)$.

Theorem 3.1 *Let Γ be the Cartesian product of two zero-divisor graph of G and H. Then $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a))$, for any $(x,r) \in V(G \times H)$.*

Proof Let $(s,r) \in N(x,a)$. Therefore, (s,r) is adjacent to (x,a). Thus, s is adjacent to x in G and $r = a$ or $s = x$ and r is adjacent to a in H. Hence, $s \in N(x)$ and $r = a$ or $s = x$ and $r \in N(a)$. It can be concluded that to $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a))$. □

Theorem 3.2 *Let G and H be two compact graphs. Then Γ the cartesian product of them is not a compact graph.*

Proof Let (x,a) and (y,b) be two arbitrary vertices not being adjacent of the graph Γ, where $(x,a) \neq (y,b)$. Therefore, x and y are not adjacent in G or $a \neq b$ in H and $x \neq y$ in G or a,b are not adjacent in H. Assume that there exists $(z,c) \in V(\Gamma)$ such that $N(x,a) \cup N(y,b) \subseteq N(z,c)$. That is,

$$\begin{aligned}(N(x) \times \{a\}) \cup (\{x\} &\times N(a)) \cup (N(y) \times \{b\}) \cup (\{y\} \times N(b))\\ &\subseteq N(z) \times \{c\}) \cup (\{z\} \times N(c)).\end{aligned}$$

Assume that $(m,a),(n,a) \in (N(x) \times \{a\})$ such that $(m,a) \in N(z) \times \{c\}$ and $(n,a) \in \{z\} \times N(c)$. Then, $m \in N(z), a = c, a \in N(c)$. Hence, $ac = 0$ and $c^2 = 0$. That is a contradiction. Therefore, $N(x) \times \{a\}$ has intersection only one of $N(z) \times \{c\}$ and $\{z\} \times N(c)$. Similary, we get this subject for $(\{x\} \times N(a)), (N(y) \times \{b\})$ and $(\{y\} \times N(b))$.

Now, suppose $N(x) \times \{a\} \subseteq N(z) \times \{c\}($ i.e., $a = c, N(x) \subseteq N(z))$. If $\{x\} \times N(a) \subseteq N(z) \times \{c\}$, we have $N(a) = \{c\}$. Hence, $ac = 0$. On the other hand $a = c$, then $c^2 = 0$. That is a contradiction. Therefore, $\{x\} \times N(a) \subseteq \{z\} \times N(c)$, that is $x = z$ and $N(a) \subseteq N(c)$. Then, $N(x) = N(z)$ and $N(a) = N(c)$.

Suppose $N(y) \times \{b,\} \subseteq N(z) \times \{c\}$. Thus, $N(y) \subseteq N(z) = N(x), b = c$. Hence, $a = b = c, N(a) = N(b) = N(c)$.

If $\{y\} \times N(b) \subseteq N(z) \times \{c\}, y \in N(z)$ and $N(b) = c$. Then, $bc = c^2 = 0$. That is a contract. Therefore, $\{y\} \times N(b) \subseteq \{c\} \times N(z)$. We get $y = z$ and $N(b) \subseteq N(c)$. It leads to $a = b = c$ and $x = y = z$. That is a contradiction. □

Corollary 3.2 *Let G and H be two compact graphs of two poset. Then, the cartesian product of them is not a graph of a poset.*

Proof Referring to the theorem above and [10], it is clear. □

§4. Tensor Product

Through this section, we assume that G and H are two zero-divisor graphs of poset P and Q with the least element 0, respectively.

Theorem 4.1 *Γ is the tensor product of the graphs G and H. Then $N(x,a) = (N(x) \times N(a))$ for any $(x,a) \in V(G \times H)$.*

Proof Assume $(s,r) \in N(x,a)$. Then, (s,r) is adjacent to (x,a). By Definition 1.2, s and x are adjacent and r and a are adjacent too. Therefore, $s \in N(x)$ and $r \in N(a)$. It leads to $N(x,a) = N(x) \times N(a)$. $\square$

§5. Strong Product

Through this section, we assume that H and K are two zero-divisor graphs of poset P and Q with the least element 0, respectively.

By Definition 1.2, Theorems 3.1 and 4.1, we conclude the following theorems.

Theorem 5.1 *Γ the strong product of two zero-divisors graphs G and H of posets. Then, if runs for any $(x,a) \in V(\Gamma)$, $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a)) \cup (N(x) \times N(a))$*

Proof By Definition 1.2, for any $(r,s) \in N(x,a)$, where $(x,a) \in V(\Gamma)$, r is adjacent to x in$G$ and $s = a$ or $r = x$ and s is adjacent to a in H or r is adjacent to x in G and s is adjacent to aiH. Therefore, $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a)) \cup (N(x) \times N(a))$. $\square$

§6. Co-normal Product

Theorem 6.1 *Γ is the co-normal product of two graphs G and H of two the posets of P and Q, respectively. Then for any $(x,a) \in V(\Gamma)$, $N(x,a) = (N(x) \times V(H)) \cup (V(H) \times N(a))$.*

Proof By Definition 1.2, if $((s,r)$ is adjacent to (x,a), s and x are adjacent in G or r, a are adjacent in H. Thus, $N(x,a) = (N(x) \times V(H)) \cup (V(H) \times N(a))$. $\square$

Theorem 6.2 *If Γ is the co-normal product of two compact graphs G and H, then Γ is a compact graph.*

Proof Let (x,a) and (y,b) not be in Γ and $(x,a) \neq (y,b)$. By referring the virtue of Definition 1.2, we get x and$y$ are not adjacent in G and a and b are not adjacent in H. Then there exist $z \in G$ and $s \in H$ such that $N(x) \cup N(y) \subseteq N(z)$ and $N(a) \cup N(b) \subseteq N(c)$. Hence,

$$\begin{aligned} N(x,a) \cup N(y,b) &= (N(x) \times N(a)) \cup (N(y) \times N(b) \\ &\subseteq (N(z) \times N(c)) \cup (N(z) \times N(c)) = N(z) \times N(c) \end{aligned} \quad \square$$

Now, we get the following corollary.

Corollary 6.3 *The co-product of two zero-divisor graphs of posets is a zero-divisor graph of a poset.*

Proof By the above theorem and [10], it is clear. □

§7. Lexicographic Product

Theorem 7.1 *Γ the lexicographic product of two zero-divisor graphs G and H of the two posets P and Q, respectively. Then, $N(x,a) = (N(x) \times V(H)) \cup (\{x\} \times N(a))$, for any $(x,a) \in V(\Gamma)$.*

Proof By Definition 1.2, assume $(s,r) \in N(x,a)$. Therefore, s and x are adjacent in G or $s = x$ in G and r and a are adjacent in H. Therefore, $N(x,a) = (N(x) \times V(H)) \cup (\{x\} \times N(a))$, for any $(x,a) \in V(\gamma)$. □

§8. Modular Product

Theorem 8.1 *Γ the Modular product of two zero-divisor graphs G and H of the two posets P and Q respectively. Then, $N(x,a) = (N(x) \times N(a)) \cup (N^c(x) \times N^c(a))$, for any $(x,a) \in V(\Gamma)$.*

Proof By Definition 1.2, assume $(s,r) \in N(x,a)$. Therefore, s and x are adjacent in G while r and a are adjacent in H or s and x are not adjacent in G whereas r and a are not adjacent in H. Thus, $N(x,a) = (N(x) \times N(a)) \cup (N^c(x) \times N^c(a))$. □

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