Linear Cyclic Snakes as Super Vertex Mean Graphs

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Abstract: A super vertex mean labeling $f$ of a $(p, q)$ - graph $G(V, E)$ is defined as an injection from $E$ to the set \{1, 2, 3, \ldots, p+q\} that induces for each vertex $v$ the label defined by the rule $f^*(v) = \text{Round} \left( \frac{\sum_{e \in E_v} f(e)}{d(v)} \right)$, where $E_v$ denotes the set of edges in $G$ that are incident at the vertex $v$, such that the set of all edge label and the induced vertex labels is \{1, 2, 3, \ldots, p+q\}. All the cycles, $C_n$, $n \geq 3$ and $n \neq 4$ are super vertex mean graphs. Our attempt in this paper is to show that all the linear cyclic snakes, including $kC_4$, are also super vertex mean graphs, even though $C_4$ is not an SVM graph. We also define the term Super Vertex Mean number of graphs.

Key Words: Super vertex mean label, Smarandachely super $H$-vertex mean labeling, linear cyclic snakes, SVM number.

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injection from $E$ to the set $\{1, 2, 3, \ldots, p + q\}$ that induces for each vertex $v$ the label defined by the rule $f'_v(v) = \text{Round} \left( \frac{\sum_{e \in E_v} f(e)}{d(v)} \right)$, where $E_v$ denotes the set of edges in $G$ that are incident at the vertex $v$, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \ldots, p + q\}$.

Furthermore, a super vertex mean labeling $f$ on $G$ is Smarandachely super $H$-vertex mean labeling if the induced vertex labels on vertex in $H'$ is $\{1, 2, 3, \ldots, p(H') + q(H')\}$ for each subgraph $H' \cong H$ in $G$, where $p(H')$ and $q(H')$ are respectively the order and the size of $H'$.

A graph that accepts super vertex mean labeling is called a super vertex mean (hereafter, SVM) graph. The following results have already been proved in [7] and [8]. We use them for our further study on the super vertex mean behavior of linear cyclic snakes. Before entering into the results, we define the term cyclic snakes.

**Definition 2.2** A $kC_n$-snake has been defined as a connected graph in which all the blocks are isomorphic to the cycle $C_n$ and the block-cut point graph is a path $P$, where $P$ is the path of minimum length that contains all the cut vertices of a $kC_n$-snake. Barrientos [13] has proved that any $kC_n$-snake is represented by a string $s_1, s_2, s_3, \ldots, s_k$ of integers of length $k - 2$, where the $i$th integer, $s_i$, on the string is the distance between $i$th and $(i + 1)$th cut vertices along the path, $P$, from one extreme and is taken from $S_n = \{1, 2, 3, \ldots, \left\lceil \frac{n}{2} \right\rceil \}$. A $kC_n$-snake is said to be linear if each integer $s_i$ of its string is equal to $\left\lceil \frac{n}{2} \right\rceil$.

**Remark 2.3** The strings obtained from both the extremes are assumed to be the same. A linear cyclic snake, $kC_n$, is obtained from $k$ copies of $C_n$ by identifying the vertex $v_{i,r+1}$ in the $i$th copy of $C_n$ at a vertex $v_{i+1,1}$ in the $(i + 1)$th copy of $C_n$, where $1 \leq i \leq k - 1$ and $n = 2r$ or $n = 2r + 1$, depending upon whether $n$ is even or odd respectively.

**Note 2.4** that $v_{i,r+1} = v_{i+1,1}$ for $1 \leq i \leq k - 1$, and we consider this vertex as $v_{i+1,1}$ throughout this paper.

§3. Known Results

We have known results listed in the following.

1. All the cycles except $C_4$ are SVM graphs([8]);
2. All odd cycles, $C_n$ can be SVM labeled as many as $\left\lceil \frac{n}{2} \right\rceil$ ways and every even cycle, $C_n$, except $C_4$ can have $\left\lceil \frac{n}{2} \right\rceil - 1$ types of SVM labelings([8]);
3. A linear triangular snake, $kC_3$ with $k$ blocks is an SVM graph([7]).

§4. Linear Cyclic Snakes of Higher Orders

We proceed to prove that other linear cyclic snakes too are super vertex mean graphs.

**Theorem 4.1** Linear quadrilateral snakes, $kC_4$ with $k \geq 2$ blocks are SVM graphs.
Proof Let $kC_4$ be a linear quadrilateral snake with $p$ vertices and $q$ edges. Then $p = 3k + 1$ and $q = 4k$. Define $f : E(kC_4) \rightarrow \{1, 2, 3, \cdots, 7k+1\}$ to be $f(u_iu_{i+1}) = 7i$ if $1 \leq i \leq k - 1$ and $7k + 1 \ i = k$. When $1 \leq i \leq k - 1$ and $k \geq 2$, 

$$f(e_{i,j}) = \begin{cases} 
1, & \text{if } i = 1, \text{ and } j = 1 \\
7i - 5, & \text{if } 2 \leq i \leq k - 1, \text{ and } j = 1 \\
7i - 1, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 2 \\
7i, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 3 \\
7i - 4, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 4.
\end{cases}$$

When $i = k$, $k \geq 2$ and $k$ is even, 

$$f(e_{i,j}) = \begin{cases} 
7k - 6, & \text{if } j = 1, \\
7k - 3, & \text{if } j = 2, \\
7k - 1, & \text{if } j = 3, \\
7k + 1, & \text{if } j = 4.
\end{cases}$$

When $i = k$, $k \geq 3$ and $k$ is odd, 

$$f(e_{i,j}) = \begin{cases} 
7k - 5, & \text{if } j = 1, \\
7k - 2, & \text{if } j = 2, \\
7k + 1, & \text{if } j = 3, \\
7k - 4, & \text{if } j = 4.
\end{cases}$$

It can be easily verified that $f$ is injective. The induced vertex labels are as follows:

When $1 \leq i \leq k - 1$ and $k \geq 2$, 

$$f^v(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
7i - 6, & \text{if } 2 \leq i \leq k - 1, \text{ and } j = 1, \\
7i - 3, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 2, \\
7i - 2, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 4.
\end{cases}$$

When $i = k$, $k \geq 2$, and $k$ is even, 

$$f^v(v_{i,j}) = \begin{cases} 
7k - 5, & \text{if } j = 1, \\
7k - 4, & \text{if } j = 2, \\
7k - 2, & \text{if } j = 3, \\
7k, & \text{if } j = 4.
\end{cases}$$
When $i = k$, $k \geq 3$, and $k$ is odd,

$$f^{v}(v_{i,j}) = \begin{cases} 
7k - 6, & \text{if } j = 1, \\
7k - 3, & \text{if } j = 2, \\
7k, & \text{if } j = 3, \\
7k - 1, & \text{if } j = 4. 
\end{cases}$$

It can be easily verified that the set of edge labels and induced vertex labels is $\{1, 2, 3, \ldots, 7k + 1\}$ in two following cases:

**Case 1.** When $k$ is even,

$$f(E) = \{1, 6, 7, 3, 9, 13, 14, 10, 16, 20, 21, 27, \ldots, 7k - 12, \\
7k - 8, 7k - 7, 7k - 11, 7k - 6, 7k - 3, 7k - 1, 7k + 1\}$$

and,

$$f^{v}(V) = \{2, 4, 8, 5, 11, 15, 12, 18, 22, 19, \ldots, 7k - 10, 7k - 5, 7k - 9, 7k - 4, 7k - 2, 7k\}.$$

Therefore,

$$f(E) \cup f^{v}(V) = \{1, 2, 3, 4, \ldots, 7k - 12, 7k - 11, 7k - 10, 7k - 9, 7k - 8, \\
7k - 7, 7k - 6, 7k - 5, 7k - 4, 7k - 3, 7k - 2, 7k - 2, 7k, 7k + 1\}.$$

**Case 2.** When $k$ is odd,

$$f(E) = \{1, 6, 7, 3, 9, 13, \ldots, 7k - 12, 7k - 8, \\
7k - 7, 7k - 11, 7k - 5, 7k - 2, 7k + 1, 7k - 4\}$$

and,

$$f^{v}(V) = \{2, 4, 8, 5, \ldots, 7k - 10, 7k - 6, 7k - 9, 7k - 3, 7k, 7k - 1\}.$$

Therefore,

$$f(E) \cup f^{v}(V) = \{1, 2, 3, 4, \ldots, 7k - 12, 7k - 11, 7k - 10, 7k - 9, 7k - 8, \\
7k - 7, 7k - 6, 7k - 5, 17k - 4, 7k - 3, 7k - 2, 7k - 2, 7k, 7k + 1\}.$$

Thus it has been proved that the labeling $f : E(kC_{4}) \rightarrow \{1, 2, 3, \ldots, 7k + 1\}$ is a super vertex mean labeling. \hfill \Box

**Example 4.2** The super vertex-mean labelings of two linear quadrilateral snakes with 4 and 3 blocks are shown in figures 1 and 2 respectively.
Theorem 4.3 All linear pentagonal snakes, \( kC_5 \) with \( k, k \geq 2 \) blocks are SVM graphs.

Proof Let \( kC_5 \) be a linear pentagonal snake with \( k, k \geq 2 \) blocks of \( C_5 \). Here \( p = 4k + 1 \), \( q = 5k \) and \( p + q = 9k + 1 \). Define \( f : E(kC_5) \to \{1, 2, 3, \ldots, 9k + 1\} \) as follows:

When \( i = 1 \),

\[
f(e_{1,j}) = \begin{cases} 
5, & \text{if } j = 1, \\
2j + 4, & \text{if } 2 \leq j \leq 3, \\
1, & \text{if } j = 4, \\
3, & \text{if } j = 5.
\end{cases}
\]

And when \( 2 \leq i \leq k \),

\[
f(e_{i,j}) = \begin{cases} 
9i - 9, & \text{if } j = 1, \\
9i + 2j - 5, & \text{if } 2 \leq j \leq 3, \\
9i + 3j - 18, & \text{if } 4 \leq j \leq 5.
\end{cases}
\]

It can be easily verified that \( f \) is injective. Then, the induced vertex labels are as follows:

When \( i = 1 \),

\[
f''(v_{i,j}) = \begin{cases} 
4, & \text{if } j = 1, \\
7, & \text{if } j = 2, \\
6, & \text{if } j = 4, \\
2, & \text{if } j = 5.
\end{cases}
\]
When \(2 \leq i \leq k\),

\[
f^v(v_{i,j}) = \begin{cases} 
9i + 2j - 9, & \text{if } 1 \leq j \leq 2 \\
9i - 2j + 6, & \text{if } 4 \leq j \leq 5 \\
9k, & \text{if } i = k, \text{ and } j = 3.
\end{cases}
\]

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is \(\{1, 2, 3, \cdots, 9k + 1\}\). Therefore, all linear pentagonal snakes \(kC_5\) are super vertex mean graphs.

**Example 4.4** A graph given in Figure 3 is an SVM labeling of a linear pentagonal snake with 3 blocks.

![Graph](image)

**Figure 3** An SVM labeling of a linear pentagonal snake, \(3C_5\).

**Theorem 4.5** All linear hexagonal snakes, \(kC_6, k \geq 2\) are super vertex mean graphs.

**Proof** Let \(kC_6\) be a hexagonal snake with \(k, k \geq 2\) blocks of \(C_6\). \(p = 5k + 1\) and \(q = 6k\) and \(p + q = 11k + 1\). Define \(f : E(G_n) \to \{1, 2, 3, \cdots, 11k + 1\}\) as follows:

\[
f(e_{i,j}) = \begin{cases} 
3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 4, \\
7, & \text{if } i = 1, \text{ and } j = 5, \\
1, & \text{if } i = 1, \text{ and } j = 6, \\
11i - 4j, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\
11i + 3j - 11, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 4, \\
11i - 8j + 37, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 6.
\end{cases}
\]
It can be easily verified that $f$ is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 
11i + 3j - 12, & \text{if } 1 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\
8, & \text{if } i = 1, \text{ and } j = 3, \\
11i - 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
11k, & \text{if } i = k, \text{ and } j = 4, \\
11i - 1, & \text{if } 1 \leq i \leq k, \text{ and } j = 5, \\
11i - 7, & \text{if } 1 \leq i \leq k, \text{ and } j = 6. 
\end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 11k + 1\}$.

Therefore, linear hexagonal snakes, all $kC_6$ with $k$ blocks of $C_6$ are super vertex mean graphs. \hfill \Box

**Example 4.6** A graph given in Figure 4 is an SVM labeling of a linear hexagonal snake with 3 blocks.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{An SVM labeling of a linear hexagonal snake, $3C_6$.}
\end{figure}

**Theorem 4.7** All linear heptagonal snakes, $kC_7, k \geq 2$ are super vertex mean graphs.

**Proof** Let $kC_7$ be a linear heptagonal snake with $k, k \geq 2$ blocks of $C_7$. $p = 6k + 1$ and $q = 7k$ and $p + q = 13k + 1$. Define $f : E(G_n) \to \{1, 2, 3, \cdots, 13k + 1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 
4j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\
30 - 4j, & \text{if } i = 1, \text{ and } 4 \leq j \leq 6, \\
1, & \text{if } i = 1, \text{ and } j = 7, \\
13i + 2j - 7, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 4, \\
13i - 13, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\
13i + 2j - 21, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7. 
\end{cases}$$
It can be easily verified that \( f \) is injective. Then, the induced vertex labels are as follows:

\[
f^v(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq 3, \\
32 - 4j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 7, \\
13i - 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
13i + 2j - 8, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3, \\
13i - 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\
13i + 3j - 29, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7, \\
13k, & \text{if } i = k, \text{ and } j = 4.
\end{cases}
\]

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is \( \{1, 2, 3, \cdots, 13k + 1\} \).

Therefore, all linear heptagonal snakes, \( kC_7 \) with \( k \) blocks of \( C_7 \) are super vertex mean graphs. \( \square \)

**Example 4.8** A graph given in Figure 5 is an SVM labeling of a linear heptagonal snake, \( 3C_7 \).

![Figure 4: A Super Vertex Mean Labeling of 3C_7 linear snake.](image)

**Theorem 4.9** Let \( kC_n \) be a linear cyclic snake with \( k, k \geq 2 \) blocks of \( C_n, n \geq 8 \) and \( n \equiv 0(\text{mod } 2) \). Then \( kC_n \) is a super vertex mean graph.

**Proof** Let \( kC_n \) be a linear cyclic snake with \( k, k \geq 2 \) blocks of \( C_n, n \geq 8 \) and \( n \equiv 0(\text{mod } 2) \). Let \( n = 2r, r \geq 4 \). Now, \( p = (n - 1)k + 1 \) and \( q = nk \) and \( p + q = (2n - 1)k + 1 \). Define
$f : E(kC_n) \to \{1, 2, 3, \cdots, (2n - 1)k + 1\}$ as follows:

\[
f(e_{i,j}) = \begin{cases} 
3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\
4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\
4n - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq n - 2, \\
7, & \text{if } i = 1, \text{ and } j = n - 1, \\
1, & \text{if } i = 1, \text{ and } j = n,
\end{cases}
\]  

\[
f(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\
4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\
4n - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq n - 1, \\
4, & \text{if } i = 1, \text{ and } j = n,
\end{cases}
\]  

And, the induced vertex labels are as follows:

\[
f(v_{i,j}) = \begin{cases} 
(2n - 1)i - 2n + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(2n - 1)i - 2n + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\
(2n - 1)i - 2n + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
(2n - 1)i - 2n + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\
(2n - 1)i + 2n - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq n - 1, \\
(2n - 1)i - 2n + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = n, \\
(2n - 1)k, & \text{if } i = k, \text{ and } j = r + 1.
\end{cases}
\]

We prove the theorem by mathematical induction on $r$, where $n = 2r, r \geq 4$. The above edge labeling function $f(e)$ and the induced vertex labeling function $f'(v)$ are expressed in terms of $r$ as follows:

\[
f(e_{i,j}) = \begin{cases} 
3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\
4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\
8r - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r - 2, \\
7, & \text{if } i = 1, \text{ and } j = 2r - 1, \\
1, & \text{if } i = 1, \text{ and } j = 2r
\end{cases}
\]
and

\[ f(e_{i,j}) = \begin{cases} (4r - 1)i - 4r + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r - 1)i - 4r + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r - 1)i - 4r + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r - 1)i - 4r + 4j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r - 1)i + 4r - 4j + 5, & \text{if } 2 \leq i \leq k, \text{ and } r + 1 \leq j \leq 2r - 1, \\ (4r - 1)(i - 1), & \text{if } i = k, \text{ and } j = 2r. \end{cases} \]

\[ f'(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\ 8r - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq 2r - 1, \\ 4, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r - 1)i - 4r + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r - 1)i - 4r + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r - 1)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r - 1)i - 4r + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r - 1)i + 4r - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq 2r - 1, \\ (4r - 1)i - 4r + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\ (4r - 1)k, & \text{if } i = k, \text{ and } j = r + 1. \end{cases} \]

We prove that the theorem is true when \( r = 4, n = 8 \).

When \( r = 4 \) the linear cyclic snake is a linear octagonal snake with \( k, k \geq 2 \) cycles of \( C_8 \).

Now, \( p = 7k + 1 \) and \( q = 8k \) and \( p+q = 15k+1 \). Define \( f : E(kC_n) \to \{1, 2, 3, \ldots, 15k+1\} \) as follows:

\[ f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 13, & \text{if } i = 1, \text{ and } j = 4, \\ 36 - 4j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 6, \\ 7, & \text{if } i = 1, \text{ and } j = 7, \\ 1, & \text{if } i = 1, \text{ and } j = 8, \\ 15i - 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ 15i - 12, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ 15i - 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ 15i - 2, & \text{if } 2 \leq i \leq k, \text{ and } j = 4, \\ 15i - 4j + 21, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 7, \\ 15i - 15, & \text{if } i = k, \text{ and } j = 8. \end{cases} \]
It can be easily verified that \(f\) is injective. The induced vertex labels are as follows:

\[
f''(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\
38 - 4j, & \text{if } i = 1, \text{ and } 6 \leq j \leq 7, \\
4, & \text{if } i = 1, \text{ and } j = 8, \\
15i - 13, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
15i - 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\
15i - 9, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
15i - 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 4, \\
15i - 4j + 23, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7, \\
15i - 11, & \text{if } 2 \leq i \leq k, \text{ and } j = 8, \\
15k, & \text{if } i = k, \text{ and } j = 5. 
\end{cases}
\]

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is \(\{1, 2, 3, \cdots, 15k + 1\}\) as follows:

\[
\begin{align*}
  f(E) &= \{3, 6, 9, 13, 16, 12, 7, 1\} \cup \\
        & \{22, 18, 24, 28, 31, 27, 23, 15\} \cup \cdots, \\
        & \{15k - 8, 15k - 12, 15k - 6, 15k - 2, 15k + 1, \\
        & 15k - 3, 15k - 7, 15k - 15\}. \\
  f''(V) &= \{2, 5, 8, 11, 14, 10, 4\} \cup \\
        & \{17, 20, 21, 26, 29, 25, 19\} \cup \cdots, \\
        & \{15k - 13, 15k - 10, 15k - 9, 15k - 4, 15k - 1, \\
        & 15k - 5, 15k - 11, 15k\}. \\
  f(E) \cup f''(V) &= \{1, 3, 6, 7, 9, 12, 13, 16\} \cup \\
                  & \{2, 4, 5, 8, 10, 11, 14\} \cup \\
                  & \{15, 18, 22, 23, 24, 27, 28, 31\} \cup \\
                  & \{17, 19, 20, 21, 25, 26, 29\} \cup \cdots, \\
                  & \{15k - 15, 15k - 12, 15k - 8, 15k - 7, 15k - 6, \\
                  & 15k - 3, 15k - 2, 15k + 1\} \cup \\
                  & \{15k - 13, 15k - 11, 15k - 10, 15k - 9, 15k - 5, \\
                  & 15k - 4, 15k - 1, 15k\}. \\
                  & = \{1, 2, 3, 4, \cdots, 29, 30, 31, \cdots, 15k - 2, 15k - 1, 15k, 15k + 1\}.
\end{align*}
\]

Thus the theorem is true when \(r = 4\).

Now we assume that the theorem is true for \(r - 1, r \geq 5\) (i.e., for \(n - 2, n \geq 10\)). In this case, \(p = (n-3)k + 1 = (2r-3)k+1\) and \(q = (n-2)k = (2r-2)k\) and \(p + q = (2n-5)k + 1 = (4r-5)k+1\).
The induction hypothesis is that the edge labeling,

\[ f : E(kC_{2r-2}) \to \{1, 2, 3, \ldots, (4r - 5)k + 1\} \]

defined as follows, is a super vertex mean labeling, where \( r \geq 5, n \geq 10, n \equiv 0(\text{mod} \ 2) \) and \( k \geq 2 \).

\[
f(e_{i,j}) = \begin{cases} 
3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\
4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r - 1, \\
8r - 4j - 4, & \text{if } i = 1, \text{ and } r \leq j \leq 2r - 4, \\
7, & \text{if } i = 1, \text{ and } j = 2r - 3, \\
1, & \text{if } i = 1, \text{ and } j = 2r - 2, \\
(4r - 5)i - 4r + 12, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(4r - 5)i - 4r + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\
(4r - 5)i - 4r + 14, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
(4r - 5)i - 4r + 4j + 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r - 1, \\
(4r - 5)i + 4r - 4j + 1, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq 2r - 3, \\
(4r - 5)(i - 1), & \text{if } i = k, \text{ and } j = 2r - 2.
\]

and the induced vertex labeling is,

\[
f^v(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\
4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r - 1, \\
8r - 4j - 2, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r - 3, \\
4, & \text{if } i = 1, \text{ and } j = 2r - 2, \\
(4r - 5)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(4r - 5)i - 4r + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\
(4r - 5)i - 4r + 11, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
(4r - 5)i - 4r + 4j, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r - 1, \\
(4r - 5)i + 4r - 4j + 3, & \text{if } 2 \leq i \leq k, \text{ and } r + 1 \leq j \leq 2r - 3, \\
(4r - 5)i - 4r + 9, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r - 2, \\
(4r - 5)k, & \text{if } i = k, \text{ and } j = r.
\]

Now we prove that the result is true for any \( r \). If we replace \( r \) with \( r + 1 \) in the above mapping we get,

\[
f(e_{i,j}) = \begin{cases} 
3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\
4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\
8r - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r - 2.
\]
Thus the theorem is proved by mathematical induction.

\[ f(e_{i,j}) = \begin{cases} 
7, & \text{if } i = 1, \text{ and } j = 2r - 1, \\
1, & \text{if } i = 1, \text{ and } j = 2r, \\
(4r - 1)i - 4r + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(4r - 1)i - 4r + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\
(4r - 1)i - 4r + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
(4r - 1)i - 4r + 4j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\
(4r - 1)i + 4r - 4j + 5, & \text{if } 2 \leq i \leq k, \text{ and } r + 1 \leq j \leq 2r - 1, \\
(4r - 1)(i - 1), & \text{if } i = k, \text{ and } j = 2r. 
\end{cases} \]

and,

\[ f(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\
4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\
8r - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq 2r - 1, \\
4, & \text{if } i = 1, \text{ and } j = 2r, \\
(4r - 1)i - 4r + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(4r - 1)i - 4r + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\
(4r - 1)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\
(4r - 1)i - 4r + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\
(4r - 1)i + 4r - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq 2r - 1, \\
(4r - 1)i - 4r + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\
(4r - 1)k, & \text{if } i = k, \text{ and } j = r + 1. 
\end{cases} \]

This is equivalent to the original labeling in terms of \( n \), which is given in the beginning of the proof, and it is clear that \( f(E) \cup f^v(V) = \{1, 2, 3, \ldots, (4r - 1)k - 1, (4r - 1)k, (4r - 1)k + 1\} \). Thus the theorem is proved by mathematical induction. \( \square \)

**Example 4.10** A graph given in Figure 6 is an SVM labeling of a linear cyclic snake \( 2C_{12} \).

![Figure 5](image_url)  
**Figure 5** A super vertex mean labeling of a linear cyclic snake \( 2C_{12} \).

**Theorem 4.11** Let \( kC_n \) be a linear cyclic snake with \( k, k \geq 2 \) blocks of \( C_n \), \( n \geq 9 \) and \( n \equiv 1(\text{mod } 4) \). Then \( kC_n \) is a super vertex mean graph.

**Proof** Let \( kC_n \) be a cyclic snake with \( k, k \geq 2 \) blocks of \( C_n \), \( n \geq 9 \) and \( n \equiv 1(\text{mod } 4) \). Let
\(n = 2r + 1, r \geq 4,\) and \(r = 2s, s \geq 2\) so that \(n = 4s + 1.\) Now \(p = (n - 1)k + 1\) and \(q = nk\) and \(p + q = (2n - 1)k + 1.\) Define \(f : E(kC_n) \rightarrow \{1, 2, 3, \cdots , (2n - 1)k + 1\}\) as follows:

\[
f(e_{i,j}) = \begin{cases} 
n, & \text{if } i = 1, \text{ and } j = 1, \\
2j + n - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq r + 1, \\
2j - n - 2, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq n, \\
(2n - 1)i - n - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(2n - 1)i - n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq r - 3, \\
(2n - 1)i - n + 2j - 1, & \text{if } 2 \leq i \leq k, \text{ and } r - 2 \leq j \leq r - 1, \\
(2n - 1)i - n + 2j, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r + 1, \\
(2n - 1)i - 8, & \text{if } 2 \leq i \leq k, n \neq 9 \text{ and } j = r + 2, \\
(2n - 1)i - 7, & \text{if } 2 \leq i \leq k, n = 9 \text{ and } j = r + 2, \\
(2n - 1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r + 3, \\
(2n - 1)i - 3n + 2j - 3, & \text{if } 2 \leq i \leq k, \text{ and } r + 4 \leq j \leq n + 1 - s, \\
(2n - 1)i - 3n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } n + 2 - s \leq j \leq n - 1, \\
(2n - 1)i - n, & \text{if } 2 \leq i \leq k, \text{ and } j = n. \\
\end{cases}
\]

and, the induced vertex labels are as follows:

When \(i = 1, \) and \(n \geq 9,\)

\[
f^v(v_{i,j}) = \begin{cases} 
n - 1, & \text{if } j = 1, \\
2j + n - 2, & \text{if } 2 \leq j \leq r, \\
n + 1, & \text{if } j = r + 2, \\
2j - 16, & \text{if } r + 3 \leq j \leq n. \\
\end{cases}
\]

and when \(2 \leq i \leq k, \) and \(n = 9,\)

\[
f^v(v_{i,j}) = \begin{cases} 
17i - 13, & \text{if } j = 1, \\
17i + 3j - 14, & \text{if } 2 \leq j \leq 4, \\
17i, & \text{if } j = 5 \text{ and } i = k, \\
17i - 3, & \text{if } j = 6, \\
22, & \text{if } j = 7, \\
19, & \text{if } j = 8, \\
23, & \text{if } j = 9. \\
\end{cases}
\]

and when \(2 \leq i \leq k, \) and \(n \geq 13,\)

\[
f^v(v_{i,j}) = \begin{cases} 
(2n - 1)i - 3r - 1, & \text{if } j = 1, \\
(2n - 1)i - n + 2j - 3, & \text{if } 2 \leq j \leq r - 3. \\
\end{cases}
\]
and

\[
f^v(v_{i,j}) = \begin{cases} 
(2n-1)i - n + 2j - 2, & \text{if } r - 2 \leq j \leq r - 1, \\
(2n-1)i - 2, & \text{if } j = r, \\
(2n-1)i, & \text{if } i = k, \text{ and } j = r + 1, \\
(2n-1)i - 3, & \text{if } j = r + 2, \\
(2n-1)i - n - 3, & \text{if } j = r + 3, \\
(2n-1)i - 3n + 2j - 4, & \text{if } r + 4 \leq j \leq n + 1 - s, \\
(2n-1)i - 3n + 2j - 3, & \text{if } n + 2 - s \leq j \leq n - 1, \\
(2n-1)i - n - 2, & \text{if } j = n.
\end{cases}
\]

We can easily prove the theorem by the technique of mathematical induction on \( s \). The remaining of the proof is left as an exercise. \( \square \)

**Theorem 4.13** Let \( kC_n \) be a linear cyclic snake with \( k, k \geq 2 \) blocks of \( C_n, n \geq 11 \) and \( n \equiv 3(\mod 4) \). Then \( kC_n \) is a super vertex mean graph.

**Proof** Let \( kC_n \) be a linear cyclic snake with \( k, k \geq 2 \) blocks of \( C_n, n \geq 11 \) and \( n \equiv 3(\mod 4) \). Let \( n = 2r + 1, r \geq 5, \) and \( r = 2s + 1, s \geq 2 \) so that \( n = 4s + 3 \). Now, \( p = (n - 1)k + 1 \) and \( q = nk \) and \( p + q = (2n - 1)k + 1 \). Define \( f : E(kC_n) \rightarrow \{1, 2, 3, \ldots, (2n - 1)k + 1\} \) as follows:

\[
f(e_{i,j}) = \begin{cases} 
4j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq r, \\
4n - 4j + 2, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r, \\
1, & \text{if } i = 1, \text{ and } j = n, \\
(2n-1)i - 2n + 2j + 7, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 3, \\
(2n-1)i - 2n + 4j + 1, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r - 1, \\
(2n-1)i - 3r + 3j - 2, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r + 1, \\
(2n-1)i + 2n - 4j + 2, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq 2r - 2, \\
(2n-1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r - 1, \\
(2n-1)i - 4n + 2j + 7, & \text{if } 2 \leq i \leq k, \text{ and } 2r \leq j \leq n.
\end{cases}
\]

and, the induced vertex labels are as follows:

\[
f^v(v_{i,j}) = \begin{cases} 
2, & \text{if } i = 1, \text{ and } j = 1, \\
4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq r, \\
4n - 4j + 4, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq n, \\
(2n-1)i - 2n + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\
(2n-1)i - 2n + 2j + 6, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3, \\
(2n-1)i - 2n + 4j - 1, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\
(2n-1)i - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r + 2
\end{cases}
\]
and

\[ f^v(v_{i,j}) = \begin{cases} 
(2n-1)i + 2n - 4j + 4, & \text{if } 2 \leq i \leq k, \text{ and } r + 3 \leq j \leq 2r - 2, \\
(2n-1)i - 2n + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r - 1, \\
(2n-1)i - 2n + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\
(2n-1)i - 2n + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r + 1, \\
(2n-1)k, & \text{if } i = k, \text{ and } j = r + 1. 
\]

We can easily prove that the above labeling is an SVM labeling of \( kC_n \), where \( k \geq 2 \) blocks of \( C_n, n \geq 11 \) and \( n \equiv 3 (mod \ 4) \), by using the technique of mathematical induction on \( s \), where \( n = 4s + 3 \). Thus the theorem can easily be proved.

\[ \Box \]

§5. Super Vertex Mean Number

The concept of super vertex mean number arises from the earlier concepts such as, mean number, super mean number etc. M.Somasundaram and R.Ponraj have introduced the term mean number of a graph [16] and they have found the mean number of many standard graphs. Later on, A.Nagarajan et.al. introduced the concept of super mean number of a graph [9] and proved the existence of it for any graph by finding out the limit values of it. Encouraged by their works we introduce this new concept which we like to name as super vertex mean number or SVM number.

**Definition 5.1** Let \( f \) be a an injective function of a \((p,q)\) - graph \( G(V,E) \) defined from \( E \) to the set \( \{1, 2, 3, \cdots , n\} \) that induces for each vertex \( v \) the label defined by the rule \( f^v(v) = \text{Round} \left( \frac{\sum_{e \in E_v} f(e)}{d(v)} \right) \), where \( E_v \) denotes the set of edges in \( G \) that are incident at the vertex \( v \). Let \( f(E) \cup f^v(V) \subseteq \{1, 2, 3, \cdots , n\} \). If \( n \) is the smallest positive integer satisfying these conditions together with the condition that all the vertex labels as well as the edge labels are distinct, then \( n \) is called the super vertex mean number (or SVM number) of the graph \( G(V,E) \), and is denoted by \( SV_m(G) \).

**Observation 5.2** It is observed that \( SV_m(G) = p + q \), for all SVM graphs \( G \) whose order is \( p \) and size is \( q \). And for other graphs \( G \), \( SV_m(G) \geq p + q + 1 \). For graphs containing an isolated vertex or a leaf, the super vertex mean number is not defined.

Therefore, for any \((p,q)\) - graph \( G \), \( p + q \leq SV_m(G) \leq \infty \). For example, the SVM number of \( C_4 \), \( SV_m(C_4) = 9 \).

§6. Conclusion

In this paper, we have proved that all the linear cyclic snakes are super vertex mean graphs. In the case of super mean labeling, the vertex analogue of SVM, it was easy to obtain a general formula for linear cyclic snakes as well as other cyclic snakes represented the string \( s_1, s_2, s_3, \cdots , s_{k-2} \), where each \( s_i \) need not be equal. This is because when we calculate the induced edge label for an edge, by finding the average of the labels of the two vertices which are
the end points of that particular edge, we need to consider only those two vertices. Therefore the average remains the same as in the case of cycles.

But for super vertex mean labeling, when we find the induced vertex labeling of the connecting vertices of a cyclic snake we have to consider four edges that are incident on those vertices to get the average. Thus it becomes pretty difficult to obtain a general formula for cyclic snakes represented the string $s_1, s_2, s_3, \ldots, s_{k-2}$, where each $s_i$ need not be equal. Another possibility emerges is that we try to explore the SVM labeling of $KC$-snakes, which is defined as a connecting graph in which each of the $k$ many blocks is isomorphic to a cycle $C_n$ for some $n$ and the block-cut point graph is a path. As in the case of $kC_n$-snakes, a $kC$-snake too can be represented by a string of integers, $s_1, s_2, \ldots, s - k - 2$. It remains still an open problem to label a $kC$-snake which has either equal $s_i$ or different $s_i$.

References

