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On the Tangent Vector Fields of Striction Curves Along the Involute and Bertrandian Frenet Ruled Surfaces

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Abstract: In this paper we consider nine special ruled surfaces associated to an involute of a curve α and its Bertrand mate α^{**} with $k_1 \neq 0$. They are called as involute Frenet ruled and Bertrandian Frenet ruled surfaces, because of their generators which are the Frenet vector fields of curve α. First we give the striction curves of all Frenet ruled surfaces. Then the striction curves of involute and Bertrandian Frenet ruled surfaces are given in terms of the Frenet apparatus of the curve α. Some results are given on the striction curves of involute and Bertrand Frenet ruled surfaces based on the tangent vector fields in E^3.

Key Words: Frenet ruled surface, involute Frenet ruled surface, Bertrandian Frenet ruled surface, evolute-involute curve, Bertrand curve pair, striction curves.

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§1. Introduction

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$ [2]. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\left\langle \alpha', v \right\rangle = 0$ where $\alpha_s = \alpha'$. The fundamental forms of the $B-scroll$ with null directrix and Cartan frame in the Minkowskian $3-space$ are examined in [5]. The properties of some ruled surfaces are also examined in $\mathbb{E}^3$ [6] , [7] ,[9] and [11]. A striction point on a ruled surface $\varphi(s,v) = \alpha(s) + v.e(s)$ is the foot of the common normal between two consecutive generators (or ruling). To illustrate the current situation, we bring here the famous example of L. K. Graves [3], so called the $B-scroll$. The special ruled surfaces $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves. The Gauss map of B-scrolls has been examined in [1]. Deriving a curve based on an other curve is one of the main subjects in geometry. Involute-evolute curves and Bertrand curves are of these kinds. An involute of a given curve is well-known concept in Euclidean $3-space$. We can say that evolute

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and involute are methods of deriving a new curve based on a given curve. The involute of a curve is called sometimes *evolvent* and evolvents play a part in the construction of gears. The evolute is the locus of the centers of osculating circles of the given planar curve [12]. Let α and α^* be the curves in Euclidean $3-space$. The tangent lines to a curve α generate a surface called the tangent surface of α. If a curve α^* is an involute of α, then by definition α is an evolute of α^*. Hence if we are given a curve α, then its evolutes are the curves whose tangent lines intersect α orthogonally. By using a similar method we produce a new ruled surface based on an other ruled surface. The differential geometric elements of the *involute $\tilde{D}$ scroll* are examined in [10]. It is well-known that if a curve is differentiable in an open interval at each point then a set of three mutually orthogonal unit vectors can be constructed. We say the set of these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curve. The set whose elements are frame vectors and curvatures of a curve α is called Frenet-Serret apparatus of the curve. Let Frenet vector fields of α be $V_1(s), V_2(s), V_3(s)$ and let first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. Then the quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are called the Frenet-Serret apparatus of the curves. If a rigid object moves along a regular curve described parametrically by $\alpha(s)$. then we know that this object has its own intrinsic coordinate system. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where curvature functions are defined by $k_1(s) = \|V_1(s)\|$, $k_2(s) = -\left\langle V_2, \dot{V}_3 \right\rangle$.

Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. If the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^* then the curve α^* is called the involute of the curve α, for $\forall s \in I$ provided that $\langle V_1, V_1^* \rangle = 0$. We can write

$$\alpha^*(s) = \alpha(s) + (c-s)V_1(s) \tag{1.1}$$

the distance between corresponding points of the involute curve in $\mathbb{E}^3$ is ([4],[8])

$$d\big(\alpha(s), \alpha^*(s)\big) = |c-s|, c = constant, \forall s \in I.$$

Theorem 1.1([4],[8]) *The Frenet vectors of the involute α^*, based on its evolute curve α are*

$$\begin{cases} V_1^* = V_2, \\ V_2^* = \dfrac{-k_1 V_1 + k_2 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \\ V_3^* = \dfrac{k_2 V_1 + k_1 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}}. \end{cases} \tag{1.2}$$

The first and the second curvatures of involute α^* are

$$k_1^* = \frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1}, \quad k_2^* = \frac{k_2' k_1 - k_1' k_2}{\lambda k_1 \left(k_1^2+k_2^2\right)} = \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)'}{\lambda k_1 \left(k_1^2+k_2^2\right)}, \tag{1.3}$$

where $(\sigma - s)k_1 > 0,\ k_1 \neq 0.$

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^{**} : I \to \mathbb{E}^3$ be two C^2- class differentiable unit speed curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^{**}(s), V_2^{**}(s), V_3^{**}(s)$ be the Frenet frames of the curves α and α^{**}, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the principal normal vector V_2^{**} of the curve α^{**}, then the pair (α, α^{**}) is called a Bertrand curve pair [4], [8]. Also α^{**} is called a Bertrand mate. If the curve α^{**} is a Bertrand mate of α then we may write

$$\alpha^{**}(s) = \alpha(s) + \lambda V_2(s) \tag{1.4}$$

If the curve α^{**} is Bertrand mate $\alpha(s)$ then we have

$$\langle V_1^{**}(s), V_1(s) \rangle = \cos\theta = constant.$$

Theorem 1.2([4],[8]) *The distance between corresponding points of the Bertrand curve pair in $\mathbb{E}^3$ is constant.*

Theorem 1.3([4]) *If the second curvature $k_2(s) \neq 0$ along a curve $\alpha(s)$ then $\alpha(s)$ is called a Bertrand curve provided that nonzero real numbers λ and β $\lambda k_1 + \beta k_2 = 1$ hold along the curve $\alpha(s)$ where $s \in I$. It follows that a circular helix is a Bertrand curve.*

Theorem 1.4([4]) *Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^{**} : I \to \mathbb{E}^3$ be two C^2- class differentiable unit speed curves and let the quantities $\{V_1, V_2, V_3, k_1, k_2\}$ and $\{V_1^{**}, V_2^{**}, V_3^{**}, k_1^{**}, k_2^{**}\}$ be Frenet-Serret apparatus of the curves α and its Bertrand mate α^{**} respectively, then*

$$\begin{cases} V_1^{**} = \dfrac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2+\beta^2}}, \\[2ex] V_2^{**} = V_2, \\[2ex] V_3^{**} = \dfrac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2+\beta^2}}; \ \lambda k_2 > 0 \end{cases} \tag{1.5}$$

The first and the second curvatures of the offset curve α^{**} are given by

$$\begin{cases} k_1^{**} = \dfrac{\beta k_1 - \lambda k_2}{\left(\lambda^2+\beta^2\right) k_2} = \dfrac{k_1 - \lambda\left(k_1^2+k_2^2\right)}{\left(\lambda^2+\beta^2\right) k_2^2}, \\ k_2^{**} = \dfrac{1}{\left(\lambda^2+\beta^2\right) k_2}. \end{cases} \tag{1.6}$$

Due to this theorem, we can write

$$\beta k_1 - \lambda k_2 = m \Longrightarrow \frac{k_2^{**}}{k_1^{**}} = \frac{1}{\beta k_1 - \lambda k_2} = \frac{1}{m},$$

$$\left(\frac{k_2^{**}}{k_1^{**}}\right)' = \frac{-m'}{m^2 k_2 \sqrt{\lambda^2+\beta^2}} \Longrightarrow \frac{ds}{ds^{**}} = \frac{1}{k_2\sqrt{\lambda^2+\beta^2}}.$$

A differentiable one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns to each $u \in I$ a point $\alpha(u) \in \mathbb{R}^3$ and a vector $X(u) \in \mathbb{R}^3, X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ depend differentiable on u. For each $u \in I$, the line L which passes through $\alpha(u)$ and is parallel to $X(u)$ is called the line of the family at u. Given a one-parameter family of lines $\{\alpha(u), X(u)\}$ the parameterized surface

$$\varphi(u,v) = \alpha(u) + v.X(u) \ \ where\ u \in I \ \ and\ v \in \mathbb{R} \tag{1.7}$$

is called the *ruled surface* generated by the family $\{\alpha(u), X(u)\}$. The lines L are called the rulings and the curve $\alpha(u)$ is called an anchor of the surface φ, [2].

Theorem 1.5([2]) *The striction point on a ruled surface $\varphi(u,v) = \alpha(u) + v.X(u)$ is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given by*

$$c(u) = \alpha(u) - \frac{\langle \alpha_u', X_u' \rangle}{\langle X_u', X_u' \rangle}.X(u) \tag{1.8}$$

where $X_u' = D_T X(u)$.

§2. On the Tangent Vector Fields of Striction Curves Along the Involute and Bertrandian Frenet Ruled Surfaces

Definition 2.1 *In the Euclidean $3-$ space, let $\alpha(s)$ be the arc length curve. The equations*

$$\begin{cases} \varphi_1(s,u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s,u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s,u_3) = \alpha(s) + u_3 V_3(s) \end{cases} \tag{2.1}$$

are the parametrization of the ruled surface which is called $V_1 - scroll$ (tangent ruled surface), $V_2 - scroll$ (normal ruled surface) and $V_3 - scroll$ (binormal ruled surface) respectively in [6].

Theorem 2.1([6]) *The striction curves of Frenet ruled surfaces are given by the following* *matrix*

$$\begin{bmatrix} c_1-\alpha \\ c_2-\alpha \\ c_3-\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1}{k_2^2+k_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Theorem 2.2 *The tangent vector fields T_1, T_2 and T_3 belonging to striction curves of Frenet ruled surface is given by*

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta\|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1k_2}{\eta\|c_2'(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

or

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where

$$a = \frac{k_2^2}{\eta\|c_2'(s)\|}, \quad b = \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|}, \quad c = \frac{k_1k_2}{\eta\|c_2'(s)\|} \text{ and } \eta = k_1^2 + k_2^2.$$

Proof It is easy to give this matrix because we have already got the following equalities

$$T_1(s) = T_3(s) = \alpha'(s) = V_1.$$

Since $c_2(s) = \alpha(s) + \dfrac{k_1}{k_1^2+k_2^2}V_2$, where $k_1^2 + k_2^2 = \eta \neq 0$, hence we have

$$\begin{aligned} c_2'(s) &= \frac{k_2^2}{\eta}V_1 + \left(\frac{k_1}{\eta}\right)' V_2 + \frac{k_1k_2}{\eta}V_3, \\ T_2(s) &= \frac{c_2'(s)}{\|c_2'(s)\|} = \frac{\eta k_2^2 V_1 + (k_1'\eta - k_1\eta')V_2 + \eta k_2 k_1 V_3}{\left(\eta^3 k_2^4 + (k_1'\eta k_1 \eta')^2\right)^{\frac{1}{2}}}. \end{aligned}$$

□

2.1 Involute Frenet Ruled Surfaces

In this subsection, first we give the tangent, normal and binormal Frenet ruled surfaces of the involute-evolute curves. Further we write their parametric equations in terms of the Frenet apparatus of the involute-evolute curves. Hence they are called *involute Frenet ruled surfaces* as in the following way.

Definition 2.2([6]) *In the Euclidean* 3*-space, let* $\alpha(s)$ *be the arc length curve. The equations*

$$\begin{aligned}
\varphi_1^*(s,v_1) &= \alpha^*(s) + v_1V_1^*(s) = \alpha(s) + (\sigma - s)V_1(s) + v_1V_2(s),\\
\varphi_2^*(s,v_2) &= \alpha^*(s) + v_2V_2^*(s) = \alpha(s) + (\sigma - s)V_1(s) + v_2\left(\frac{-k_1V_1 + k_2V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right),\\
\varphi_3^*(s,v_3) &= \alpha^*(s) + v_3V_3^*(s) = \alpha(s) + (\sigma - s)V_1(s) + v_3\left(\frac{k_2V_1 + k_1V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right)
\end{aligned}$$

are the parametrization of the ruled surfaces which are called involute tangent ruled surface, involute normal ruled surface and involute binormal ruled surface, respectively.

We can deduce from Theorem 2.1 striction curves of the involute Frenet ruled surfaces are given by the following matrix

$$\begin{bmatrix} c_1^* - \alpha^* \\ c_2^* - \alpha^* \\ c_3^* - \alpha^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^*}{k_1^{*2}+k_2^{*2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

It is easy to give the following matrix for the striction curves of four Frenet ruled surfaces along the *involute* curve α^*.

$$\begin{aligned}
c_1^*(s) &= c_3^*(s) = \alpha^*(s),\\
c_2^*(s) &= \alpha^*(s) + \frac{k_1^*}{k_1^{*2}+k_2^{*2}}V_2^*(s).
\end{aligned}$$

Also we can write explicit equations of the striction curves on involute Frenet ruled surfaces in terms of Frenet apparatus of an evolute curve α.

Theorem 2.3 *The equations of the striction curves on involute Frenet ruled surfaces in terms of Frenet apparatus of an evolute curve* α *are given by*

$$\begin{bmatrix} c_1^* - \alpha \\ c_2^* - \alpha \\ c_3^* - \alpha \end{bmatrix} = \begin{bmatrix} \sigma - s & 0 & 0 \\ (\sigma - s)\left(1 - \frac{k_1^2}{(k_1^2+k_2^2)(1+m)}\right) & 0 & \dfrac{(\sigma - s)k_1k_2}{(k_1^2+k_2^2)(1+m)} \\ \sigma - s & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Theorem 2.4 *The tangent vector fields* ${T_1}^*, {T_2}^*, {T_3}^*$ *of striction curves belonging to an involute Frenet ruled surface in terms of Frenet apparatus by themselves are given by*

$$[T^*] = \begin{bmatrix} {T_1}^* \\ {T_2}^* \\ {T_3}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} {V_1}^* \\ {V_2}^* \\ {V_3}^* \end{bmatrix}.$$

$$a^* = \frac{k_2^{*2}}{\eta^* \left\| c_2^{*\prime}(s) \right\|},\ \ b^* = \frac{\left(\frac{k_1^*}{\eta^*}\right)^{\prime}}{\left\| c_2^{*\prime}(s) \right\|},\ \ c^* = \frac{k_1^* k_2^*}{\eta^* \left\| c_2^{*\prime}(s) \right\|},\ \ \eta^* = k_1^{*2} + k_2^{*2},\ \ \mu^* = \left(\frac{k_2^*}{k_1^*}\right)^{\prime}.$$

2.2 Bertrandian Frenet ruled surfaces

In this subsection, first we give the tangent, normal and binormal Frenet ruled surfaces of the Bertrand mate α^{**}. Further we write their parametric equations in terms of the Frenet apparatus of the Bertrand curve α. Hence they are called *Bertrandian Frenet ruled surfaces* as in the following way.

Definition 2.3([6]) *In the Euclidean $3-space$, let $\alpha(s)$ be the arc length curve. The equations*

$$\begin{aligned}
\varphi_1^{**}(s, w_1) &= \alpha^{**}(s) + w_1 V_1^{**}(s) = \alpha + \lambda V_2 + w_1 \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}}, \\
\varphi_2^{**}(s, w_2) &= \alpha^{**}(s) + w_2 V_2^{**}(s) = \alpha + (\lambda + w_2) V_2, \\
\varphi_3^{**}(s, w_3) &= \alpha^{**}(s) + w_3 V_3^{**}(s) = \alpha + \lambda V_2 + w_3 \left(\frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}}\right),
\end{aligned} \tag{2.2}$$

are the parametrization of the ruled surfaces which are called Bertrandian tangent ruled surface, Bertrandian normal ruled surface and Bertrandian binormal ruled surface, respectively.

We can also deduce from Theorem 2.1 the striction curves of Bertrand Frenet ruled surfaces are given by the following matrix

$$\begin{bmatrix} c_1^{**} - \alpha^{**} \\ c_2^{**} - \alpha^{**} \\ c_3^{**} - \alpha^{**} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^{**}}{k_1^{**2} + k_2^{**2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{bmatrix}.$$

It is easy to give the following matrix for the striction curves belonging to Bertrand Frenet ruled surfaces

$$\begin{aligned}
c_1^{**}(s) &= c_3^{**}(s) = \alpha^{**}(s) \\
c_2^{**}(s) &= \alpha^{**}(s) + \frac{k_1^{**}}{k_1^{**2} + k_2^{**2}} V_2^{**}(s)
\end{aligned}$$

Theorem 2.5 *The equations of the striction curves on Bertrandian Frenet ruled surfaces in terms of Frenet apparatus of curve α*

$$\begin{bmatrix} c_1^{**} - \alpha \\ c_2^{**} - \alpha \\ c_3^{**} - \alpha \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & \left(\lambda + \frac{m(\lambda^2 + \beta^2) k_2}{(m^2+1)}\right) & 0 \\ 0 & \lambda & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Proof Since the equations of the striction curves on Bertrandian Frenet ruled surfaces in terms of Frenet apparatus of curve α are

$$c_1^{**}(s) = c_3^{**}(s) = \alpha^{**}(s) = \alpha(s) + \lambda V_2(s)$$

the first and the second curvatures of the curve α^{**} are given by $k_1^{**} = \dfrac{\beta k_1 - \lambda k_2}{(\lambda^2+\beta^2)\,k_2}$ and $k_2^{**} = \dfrac{1}{(\lambda^2+\beta^2)\,k_2}$. Also $k_2 k_2^{**} = \dfrac{1}{(\lambda^2+\beta^2)}$ and

$$c_2^{**}(s) = \alpha^{**}(s) + \frac{k_1^{**}}{k_1^{**2}+k_2^{**2}} V_2^{**}(s) = \alpha + \left(\lambda + \frac{(\lambda^2+\beta^2)\,k_2}{\left((\beta k_1 - \lambda k_2)^2 + 1\right)}\right) V_2. \qquad \square$$

Theorem 2.6 *The tangent vector fields* T_1^{**}, T_2^{**} *and* T_3^{**} *of striction curves belonging to Bertrandian Frenet ruled surface are given by*

$$\begin{bmatrix} T_1^{**} \\ T_2^{**} \\ T_3^{**} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^{**} & b^{**} & c^{**} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{bmatrix}$$

where

$$a^{**} = \frac{k_2^{**2}}{\eta^{**}\left\|c_2^{**\prime}(s)\right\|},\ \ b^{**} = \frac{\left(\frac{k_1^{**}}{\eta^{**}}\right)'}{\left\|c_2^{**\prime}(s)\right\|},\ \ c^{**} = \frac{k_1^{**}k_2^{**}}{\eta^{**}\left\|c_2^{**\prime}(s)\right\|}\ \ \text{and}\ \ \eta^{**} = k_1^{**2} + k_2^{**2}.$$

Theorem 2.7 *The product of tangent vector fields* ${T_1}^*, {T_2}^*, {T_3}^*$ *and tangent vector fields* ${T_1}^{**}, {T_2}^{**}, {T_3}^{**}$ *of striction curves on an involute and Bertrandian Frenet ruled surface respectively, are given by*

$$[T^*]\,[T^{**}]^{\mathbf{T}} = A\begin{bmatrix} 0 & Ab^{**} & 0 \\ B & a^{**}B + b^{**}a^*A + c^{**}C & B \\ 0 & b^{**}A & 0 \end{bmatrix}$$

where the coefficients are

$$A = \sqrt{(\lambda^2+\beta^2)({k_1}^2+{k_2}^2)},\ \ B = b^*(-\beta k_1 + \lambda k_2) + c^*,\ \ C = b^* + c^*(-\lambda k_2 + \beta k_1).$$

Proof Let $[T^*] = [A^*]\,[V^*]$ and $[T^{**}] = [A^{**}]\,[V^{**}]$ be given. By using the properties of a matrix following result can be obtained:

$$
\begin{aligned}
[T^*]\,[T^{**}]^{\mathbf{T}} &= [A^*]\,[V^*]\,([A^{**}]\,[V^{**}])^{\mathbf{T}} \\
&= [A^*]\left([V^*]\,[V^{**}]^{\mathbf{T}}\right)[A^{**}]^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_1{}^* \\ V_2{}^* \\ V_3{}^* \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ a^{**} & b^{**} & c^{**} \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_1{}^{**} \\ V_2{}^{**} \\ V_3{}^{**} \end{bmatrix}\right)^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_1{}^* \\ V_2{}^* \\ V_3{}^* \end{bmatrix}\begin{bmatrix} V_1{}^{**} \\ V_2{}^{**} \\ V_3{}^{**} \end{bmatrix}^{\mathbf{T}}\begin{bmatrix} 1 & 0 & 0 \\ a^{**} & b^{**} & c^{**} \\ 1 & 0 & 0 \end{bmatrix}^{\mathbf{T}} \\
&= A\begin{bmatrix} 0 & b^{**}A & 0 \\ B & a^{**}B + b^{**}a^*A + c^{**}C & B \\ 0 & b^{**}A & 0 \end{bmatrix}. \qquad \square
\end{aligned}
$$

As a result of Theorem 2.1 we can write that in the Euclidean $3-space$, the position of the unit tangent vector field T_1^*, T_2^*, T_3^* and $T_1^{**}, T_2^{**}, T_3^{**}$ of striction curves belonging to ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*$ and $\varphi_1^{**}, \varphi_2^{**}, \varphi_3^{**}$ respectively, along the curve α^* and α^{**}, can be expressed by the following equations

$$
[T^*]\,[T^{**}]^T = \begin{bmatrix} \langle T_1^*, T_1^{**}\rangle & \langle T_1^*, T_2^{**}\rangle & \langle T_1^*, T_3^{**}\rangle \\ \langle T_2^*, T_1^{**}\rangle & \langle T_2^*, T_2^{**}\rangle & \langle T_2^*, T_3^{**}\rangle \\ \langle T_3^*, T_1^{**}\rangle & \langle T_3^*, T_2^{**}\rangle & \langle T_3^*, T_3^{**}\rangle \end{bmatrix},
$$

here $[T^{**}]^T$ is the transpose matrix of $[T^{**}]$.

Hence we may write that, there are four tangent vector fields on striction curves which are perpendicular to each other, for the involute and Bertrandian Frenet ruled surfaces given above. Since $\langle T_1^*, T_1^{**}\rangle = \langle T_1^*, T_3^{**}\rangle = \langle T_3^*, T_1^{**}\rangle = \langle T_3^*, T_3^{**}\rangle = 0$, it is trivial.

Theorem 2.8 (i) *The tangent vector fields of striction curves on an involute tangent and Bertrandian normal ruled surfaces are perpendicular under the condition*

$$
\left[\frac{(\beta k_1 - \lambda k_2)(\lambda^2+\beta^2)k_2}{(\beta k_1 - \lambda k_2)^2 + 1}\right]' = 0, \;\; \lambda^2 = -\beta^2 \;\; or \;\; k_1{}^2 = -k_2{}^2.
$$

(ii) *The tangent vector fields of striction curves on an involute binormal and Bertrandian normal ruled surfaces are perpendicular under the condition*

$$
\left[\frac{(\beta k_1 - \lambda k_2)(\lambda^2+\beta^2)k_2}{(\beta k_1 - \lambda k_2)^2 + 1}\right]' = 0, \;\; \lambda^2 = -\beta^2 \;\; or \;\; k_1{}^2 = -k_2{}^2.
$$

Proof (i) Since $\langle T_1^*, T_2^{**}\rangle = b^{**}A$ and $\langle T_1^*, T_2^{**}\rangle = 0$

$$\begin{aligned} b^{**}A &= 0 \\ \left[\frac{(\beta k_1 - \lambda k_2)(\lambda^2+\beta^2)k_2}{(\beta k_1-\lambda k_2)^2+1}\right]' \sqrt{(\lambda^2+\beta^2)({k_1}^2+{k_2}^2)} &= 0 \end{aligned}$$

$$\left[\frac{(\beta k_1 - \lambda k_2)(\lambda^2+\beta^2)k_2}{(\beta k_1-\lambda k_2)^2+1}\right]' = 0 \ \ or \ \ \sqrt{(\lambda^2+\beta^2)({k_1}^2+{k_2}^2)} = 0,$$

this completes the proof.

(ii) Since $\langle T_1^*, T_2^{**}\rangle = \langle T_3^*, T_2^{**}\rangle = b^{**}A$, the proof is trivial. □

Theorem 2.9 (i) *The tangent vector fields of striction curves on an involute normal and Bertrandian tangent ruled surfaces are perpendicular under the condition*

$$-\beta k_1 + \lambda k_2 = \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3 + k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'}.$$

(ii) *The tangent vector fields of striction curves on an involute normal and Bertrandian binormal ruled surfaces are perpendicular under the condition*

$$-\beta k_1 + \lambda k_2 = \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3 + k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'}.$$

Proof (i) Since $\langle T_2^*, T_1^{**}\rangle = B = b^*(-\beta k_1 + \lambda k_2) + c^*$ and $\langle T_2^*, T_1^{**}\rangle = 0$

$$\begin{aligned} B = b^*(-\beta k_1 + \lambda k_2) + c^* &= 0 \\ \beta k_1 - \lambda k_2 + \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3 + k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'} &= 0 \end{aligned}$$

$$-\beta k_1 + \lambda k_2 = \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3 + k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'},$$

this completes the proof.

(ii) Since $\langle T_2^*, T_1^{**}\rangle = \langle T_2^*, T_3^{**}\rangle = B = b^*(-\beta k_1 + \lambda k_2) + c^*$, the proof is trivial. □

Corollary 2.1 *The inner product between tangent vector fields of striction curves on an involute* *normal and Bertrandian normal ruled surfaces of the* (α^*, α^{**}) *is*

$$\langle T_2^*, T_2^{**} \rangle = a^{**}B + b^{**}a^*A + c^{**}C.$$

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