Ricci Soliton and Conformal Ricci Soliton in Lorentzian $\beta$-Kenmotsu Manifold

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Abstract: In this paper we have studied quasi conformal curvature tensor, Ricci tensor, projective curvature tensor, pseudo projective curvature tensor in Lorentzian $\beta$-Kenmotsu manifold admitting Ricci soliton and conformal Ricci soliton.

Key Words: Trans-Sasakian manifold, $\beta$-Kenmotsu manifold, Lorentzian $\beta$-Kenmotsu manifold, Ricci soliton, conformal Ricci flow.


§1. Introduction

Hamilton started the study of Ricci flow [12] in 1982 and proved its existence. This concept was developed to answer Thurston’s geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [11] classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman also did an excellent work on Ricci flow [15], [16].

The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S$$

on a compact Riemannian manifold $M$ with Riemannian metric $g$. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling. Ramesh Sharma [18], M. M. Tripathi [19], Bejan, Crasmareanu [4] studied Ricci soliton in contact metric manifolds also. The Ricci soliton equation is given by

$$\mathcal{L}_X g + 2S + 2\lambda g = 0,$$

where $\mathcal{L}_X$ is the Lie derivative, $S$ is Ricci tensor, $g$ is Riemannian metric, $X$ is a vector field and $\lambda$ is a scalar. The Ricci soliton is said to be shrinking, steady and expanding according as

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\( \lambda \) is negative, zero and positive respectively.

In 2005, A.E. Fischer [10] introduced the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation scalar curvature \( R \) is considered as constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. The conformal Ricci flow equation on \( M \) where \( M \) is considered as a smooth closed connected oriented \( n \)-manifold \((n > 3)\), is defined by the equation [10]

\[
\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg
\]

and \( r = -1 \), where \( p \) is a scalar non-dynamical field (time dependent scalar field), \( r \) is the scalar curvature of the manifold and \( n \) is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton and the equation is as follows

\[
\mathcal{L}Xg + 2S = [2\lambda - (p + \frac{2}{n})]g.
\]

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

An almost contact metric structure \((\phi, \xi, \eta, g)\) on a manifold \( M \) is called a trans-Sasakian structure [14] if the product manifold belongs to the class \( W_4 \) where \( W_4 \) is a class of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [6]. A trans-Sasakian structure of type \((0, 0), (0, \beta)\) and \((\alpha, 0)\) are cosymplectic [5], \( \beta \)-Kenmotsu [13], and \( \alpha \)-Sasakian [13], respectively.

§2. Preliminaries

A differentiable manifold of dimension \( n \) is called Lorentzian Kenmotsu manifold [2] if it admits a \((1, 1)\) tensor field \( \phi \), a covariant vector field \( \xi \), a 1-form \( \eta \) and Lorentzian metric \( g \) which satisfy on \( M \) respectively such that

\[
\phi^2 X = X + \eta(X)\xi, g(X, \xi) = \eta(X),
\]

\[
\eta(\xi) = -1, \eta(\phi X) = 0, \phi \xi = 0,
\]

\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\]

for all \( X, Y \in \chi(M) \).

If Lorentzian Kenmotsu manifold \( M \) satisfies

\[
\nabla_X \xi = \beta[X - \eta(X)\xi], (\nabla_X \phi) Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),
\]

(2.4)
\[(\nabla_X \eta)Y = \alpha g(\phi X, Y),\]  

(2.5)

where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\). Then the manifold \(M\) is called Lorentzian \(\beta\)-Kenmotsu manifold.

Furthermore, on an Lorentzian \(\beta\)-Kenmotsu manifold \(M\) the following relations hold [1], \[17\]:

\[
\eta(R(X, Y)Z) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)],
\]

(2.6)

\[
R(\xi, X)Y = \beta^2[\eta(\xi)X - g(X, Y)\xi],
\]

(2.7)

\[
R(X, Y)\xi = \beta^2[\eta(\xi)X - \eta(Y)X],
\]

(2.8)

\[
S(X, \xi) = -(n - 1)\beta^2 \eta(X),
\]

(2.9)

\[
Q\xi = -(n - 1)\beta^2 \xi,
\]

(2.10)

\[
S(\xi, \xi) = (n - 1)\beta^2,
\]

(2.11)

where \(\beta\) is some constant, \(R\) is the Riemannian curvature tensor, \(S\) is the Ricci tensor and \(Q\) is the Ricci operator given by \(S(X, Y) = g(QX, Y)\) for all \(X, Y \in \chi(M)\).

Now from definition of Lie derivative we have

\[
(\mathcal{L}_\xi g)(X, Y) = (\nabla_\xi g)(X, Y) + g(\beta[X - \eta(X)\xi], Y) + g(X, \beta[Y - \eta(Y)\xi])
\]

\[
= 2\beta g(X, Y) - 2\beta \eta(X)\eta(Y).
\]

(2.12)

Applying Ricci soliton equation (1.2) in (2.12) we get

\[
S(X, Y) = \frac{1}{2}[-2\lambda g(X, Y)] - \frac{1}{2}[2\beta g(X, Y) - 2\beta \eta(X)\eta(Y)]
\]

\[
= -\lambda g(X, Y) - \beta g(X, Y) + \beta \eta(X)\eta(Y)
\]

\[
= \tilde{A}g(X, Y) + \beta \eta(X)\eta(Y),
\]

(2.13)

where \(\tilde{A} = (-\lambda - \beta)\), which shows that the manifold is \(\eta\)-Einstein.

Also

\[
QX = \tilde{A}X + \beta \eta(X)\xi,
\]

(2.14)

\[
S(X, \xi) = (\tilde{A} + \beta) \eta(X) = A\eta(X).
\]

(2.15)

If we put \(X = Y = e_i\) in (2.13) where \(\{e_i\}\) is the orthonormal basis of the tangent space \(TM\) where \(TM\) is a tangent bundle of \(M\) and summing over \(i\), we get

\[
R(g) = \tilde{A}n + \beta.
\]

**Proposition 2.1** A Lorentzian \(\beta\)-Kenmotsu manifold admitting Ricci soliton is \(\eta\)-Einstein.
Again applying conformal Ricci soliton (1.4) in (2.12) we get

\[ S(X,Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{n})]g(X,Y) - \frac{1}{2}[2\beta g(X,Y) - 2\beta \eta(X)\eta(Y)] \]
\[ = \hat{B}g(X,Y) + \beta \eta(X)\eta(Y), \]  
(2.16)

where
\[ \hat{B} = \frac{1}{2}[2\lambda - (p + \frac{2}{n})] - \beta, \]
(2.17)

which also shows that the manifold is \( \eta \)-Einstein.

Also
\[ QX = \hat{B}X + \beta \eta(X)\xi, \]
(2.18)
\[ S(X, \xi) = (\hat{B} + \beta)\eta(X) = B\eta(X). \]
(2.19)

If we put \( X = Y = e_i \) in (2.16) where \( \{e_i\} \) is the orthonormal basis of the tangent space \( TM \) where \( TM \) is a tangent bundle of \( M \) and summing over \( i \), we get
\[ r = \hat{B}n + \beta. \]

For conformal Ricci soliton \( r(g) = -1 \). So
\[ -1 = \hat{B}n + \beta \]

which gives \( B = \frac{1}{n}(-\beta - 1) \).

Comparing the values of \( B \) from (2.17) with the above equation we get
\[ \lambda = \frac{1}{n}(\beta(n-1) - 1) + \frac{1}{2}(p + \frac{2}{n}) \]

**Proposition 2.2** A Lorentzian \( \beta \)-Kenmotsu manifold admitting conformal Ricci soliton is \( \eta \)-Einstein and the value of the scalar
\[ \lambda = \frac{1}{n}(\beta(n-1) - 1) + \frac{1}{2}(p + \frac{2}{n}). \]

§3. Lorentzian \( \beta \)-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and \( R(\xi, X).\tilde{C} = 0 \)

Let \( M \) be a \( n \) dimensional Lorentzian \( \beta \)-Kenmotsu manifold admitting Ricci soliton \( (g, V, \lambda) \). Quasi conformal curvature tensor \( \tilde{C} \) on \( M \) is defined by

\[ \tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \]
\[ -\frac{r}{2n + 1}[\frac{a}{2n} + 2b][g(Y, Z)X - g(X, Z)Y], \]
(3.1)

where \( r \) is scalar curvature.
Putting $Z = \xi$ in (3.1) we have
\[
\tilde{C}(X, Y)\xi = aR(X, Y)\xi + b[S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY] \\
- \frac{r}{2n + 1} \left( \frac{a}{2n} + 2b \right) \left[ g(Y, \xi)X - g(X, \xi)Y \right].
\]
(3.2)

Using (2.1), (2.8), (2.14), (2.15) in (3.2) we get
\[
\tilde{C}(X, Y)\xi = [-a\beta^2 + Ab + \dot{A}b - \frac{r}{2n + 1} \left( \frac{a}{2n} + 2b \right)] (\eta(Y)X - \eta(X)Y).
\]

Let
\[
D = -a\beta^2 + Ab + \dot{A}b - \frac{r}{2n + 1} \left( \frac{a}{2n} + 2b \right),
\]
so we have
\[
\tilde{C}(X, Y)\xi = D(\eta(Y)X - \eta(X)Y).
\]
(3.3)

Taking inner product with $Z$ in (3.3) we get
\[
-\eta(\tilde{C}(X, Y)Z) = D[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)].
\]
(3.4)

Now we consider that the Lorentzian $\beta$-Kenmotsu manifold $M$ which admits Ricci soliton is quasi conformally semi symmetric i.e. $R(\xi, X)\tilde{C} = 0$ holds in $M$, which implies
\[
R(\xi, X)(\tilde{C}(Y, Z)W) - \tilde{C}(R(\xi, X)Y, Z)W - \tilde{C}(Y, R(\xi, X)Z)W - \tilde{C}(Y, Z)R(\xi, X)W = 0,
\]
(3.5)

for all vector fields $X, Y, Z, W$ on $M$.

Using (2.7) in (3.5) and putting $W = \xi$ we get
\[
\eta(\tilde{C}(Y, Z)\xi)X - g(X, \tilde{C}(Y, Z)\xi) - \eta(Y)\tilde{C}(X, Z)\xi + g(X, Y)\tilde{C}(\xi, Z)\xi \\
- \eta(Z)\tilde{C}(Y, X)\xi + g(X, Z)\tilde{C}(Y, \xi)\xi - \eta(\xi)\tilde{C}(Y, Z)X + g(X, \xi)\tilde{C}(Y, Z)\xi = 0.
\]
(3.6)

Taking inner product with $\xi$ in (3.6) and using (2.2), (3.3) we obtain
\[
g(X, \tilde{C}(Y, Z)\xi) + \eta(\tilde{C}(Y, Z)X) = 0.
\]
(3.7)

Putting $Z = \xi$ in (3.7) and using (3.3) we get
\[
-Dg(X, Y) - D\eta(X)\eta(Y) + \eta(\tilde{C}(Y, Z)X) = 0.
\]
(3.8)

Now from (3.1) we can write
\[
\tilde{C}(Y, \xi)X = aR(Y, \xi)X + b[S(\xi, X)Y - S(Y, X)\xi + g(\xi, X)QY - g(Y, X)Q\xi] \\
- \frac{r}{2n + 1} \left( \frac{a}{2n} + 2b \right) [g(\xi, X)Y - g(Y, X)\xi].
\]
(3.9)
Taking inner product with \( \xi \) and using (2.2), (2.7), (2.9), (2.10) in (3.9) we get

\[
\eta(\tilde{\mathcal{C}}(Y, \xi)X) = a\eta(\beta^2(g(X,Y)\xi - \eta(X)Y)) + b[A\eta(X)\eta(Y) + S(X,Y) + \eta(X)(\dot{A}\eta(Y) - \beta\eta(Y)) - g(X,Y)(-\dot{A} + \beta)] - \left[\frac{r}{2n+1}\frac{a}{2n} + 2b][\eta(X)\eta(Y) + g(X,Y)]\right].
\]

After a long simplification we have

\[
\eta(\tilde{\mathcal{C}}(Y, \xi)X) = g(X,Y)[\dot{A}b - b\beta - a\beta^2 - \left[\frac{r}{2n+1}\frac{a}{2n} + 2b]\right]
+ \eta(X)\eta(Y)[2\dot{A}b - a\beta^2 - \left[\frac{r}{2n+1}\frac{a}{2n} + 2b]\right] + bS(X,Y).
\]

Putting (3.10) in (3.5) we get

\[
\rho g(X,Y) + \sigma \eta(X)\eta(Y) = S(X,Y),
\]

where

\[
\rho = \frac{1}{b}[D + b\beta + a\beta^2 - \dot{A}b + \left[\frac{r}{2n+1}\frac{a}{2n} + 2b]\right]
\]

and

\[
\sigma = \frac{1}{b}[D + a\beta^2 - 2\dot{A}b + \left[\frac{r}{2n+1}\frac{a}{2n} + 2b]\right].
\]

So from (3.11) we conclude that the manifold becomes \( \eta \)-Einstein manifold. Thus we can write the following theorem:

**Theorem 3.1** If a Lorentzian \( \beta \)-Kenmotsu manifold admits Ricci soliton and is quasi conformally semi symmetric i.e. \( R(\xi, X)\tilde{\mathcal{C}} = 0 \), then the manifold is \( \eta \)-Einstein manifold where \( \tilde{\mathcal{C}} \) is quasi conformal curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

If a Lorentzian \( \beta \)-Kenmotsu manifold admits conformal Ricci soliton then after a brief calculation we can also establish that the manifold becomes \( \eta \)-Einstein, only the values of constants \( \rho, \sigma \) will be changed which would not hamper our main result.

Hence we can state the following theorem:

**Theorem 3.2** A Lorentzian \( \beta \)-Kenmotsu manifold admitting conformal Ricci soliton and is quasi conformally semi symmetric i.e. \( R(\xi, X)\tilde{\mathcal{C}} = 0 \), then the manifold is \( \eta \)-Einstein manifold where \( \tilde{\mathcal{C}} \) is quasi conformal curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

**§4. Lorentzian \( \beta \)-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and \( R(\xi, X).S = 0 \)**

Let \( M \) be a \( n \) dimensional Lorentzian \( \beta \)-Kenmotsu manifold admitting Ricci soliton \( (g, V, \lambda) \). Now we consider that the tensor derivative of \( S \) by \( R(\xi, X) \) is zero i.e. \( R(\xi, X).S = 0 \). Then the
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Lorentzian \( \beta \)-Kenmotsu manifold admitting Ricci soliton is Ricci semi symmetric which implies

\[
S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{4.1}
\]

Using (2.13) in (4.1) we get

\[
\hat{A}g(R(\xi, X)Y, Z) + \beta \eta(R(\xi, X)Y)\eta(Z) + \hat{A}g(Y, R(\xi, X)Z) + \beta \eta(Y)\eta(R(\xi, X)Z) = 0. \tag{4.2}
\]

Using (2.7) in (4.2) we get

\[
\hat{A}g(\beta^2[\eta(Y) X - g(X, Y)\xi], Z) + \hat{A}g(Y, \beta^2[\eta(Z) X - g(X, Z)\xi]) + \beta \eta(\beta^2[\eta(Y) X - g(X, Y)\xi]) = 0. \tag{4.3}
\]

Using (2.2) in (4.3) we have

\[
\hat{A}\beta^2\eta(Y) g(X, Z) - \hat{A}\beta^2\eta(Z) g(X, Y) + \hat{A}\beta^2\eta(Z) g(X, Y) - \hat{A}\beta^2\eta(Y) g(X, Z) + \beta^3\eta(Y) \eta(Z) + \beta^3\eta(Y) \eta(X) \eta(Z) + \beta^3 g(X, Z) \eta(Y) = 0. \tag{4.4}
\]

Putting \( Z = \xi \) in (4.4) and using (2.2) we obtain

\[
g(X, Y) = -\eta(X)\eta(Y).
\]

Hence we can state the following theorem:

**Theorem 4.1** If a Lorentzian \( \beta \)-Kenmotsu manifold admits Ricci soliton and is Ricci semi symmetric i.e. \( R(\xi, X).S = 0 \), then \( g(X, Y) = -\eta(X)\eta(Y) \) where \( S \) is Ricci tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

If a Lorentzian \( \beta \)-Kenmotsu manifold admits conformal Ricci soliton then by similar calculation we can obtain the same result. Hence we can state the following theorem:

**Theorem 4.2** A Lorentzian \( \beta \)-Kenmotsu manifold admitting conformal Ricci soliton and is Ricci semi symmetric i.e. \( R(\xi, X).S = 0 \), then \( g(X, Y) = -\eta(X)\eta(Y) \) where \( S \) is Ricci tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

\[ \text{§5. Lorentzian } \beta \text{-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and } R(\xi, X).P = 0 \]

Let \( M \) be a \( n \) dimensional Lorentzian \( \beta \)-Kenmotsu manifold admitting Ricci soliton \((g, V, \lambda)\). The projective curvature tensor \( P \) on \( M \) is defined by

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n} [S(Y, Z)X - S(X, Z)Y]. \tag{5.1}
\]

Here we consider that the manifold is projectively semi symmetric i.e. \( R(\xi, X).P = 0 \) holds.
So
\[ R(\xi, X)(P(Y, Z)W) - P(R(\xi, X)Y, Z)W - P(Y, R(\xi, X)Z)W - P(Y, Z)R(\xi, X)W = 0, \quad (5.2) \]
for all vector fields \( X, Y, Z, W \) on \( M \).

Using (2.7) and putting \( Z = \xi \) in (5.2) we have
\[
\eta(P(Y, \xi)W)X - g(X, P(Y, \xi)W)\xi - \eta(Y)P(\xi, \xi)W + g(X, Y)P(\xi, \xi)W
\]
\[-\eta(\xi)P(Y, \xi)W + g(\xi, \xi)P(Y, \xi)W - \eta(\xi)P(Y, \xi)X + g(\xi, \xi)P(Y, \xi)X = 0. \quad (5.3)\]

Now from (5.1) we can write
\[ P(X, \xi)Z = R(X, \xi)Z - \frac{1}{n-1}[S(\xi, Z)X - S(X, Z)\xi]. \quad (5.4) \]

Using (2.7), (2.15) in (5.4) we get
\[ P(X, \xi)Z = \beta^2 g(X, Z)\xi + \frac{1}{n-1} S(X, Z)\xi + (\frac{A}{n-1} - \beta^2)\eta(Z)X. \quad (5.5) \]

Putting (5.5) and \( W = \xi \) in (5.3) and after a long calculation we get
\[
\frac{1}{n-1} S(X, Y)\xi + (\frac{A}{n-1} + \beta^2)\eta(Y)X - \frac{A}{n-1} g(X, Y)\xi
\]
\[-(\frac{A}{n-1} + \beta^2)\eta(Y)X = 0. \quad (5.6)\]

Taking inner product with \( \xi \) in (5.6) we obtain
\[ S(X, Y) = -Ag(X, Y), \]
which clearly shows that the manifold is an Einstein manifold.

Thus we can conclude the following theorem:

**Theorem 5.1** If a Lorentzian \( \beta \)-Kenmotsu manifold admits Ricci soliton and is projectively semi symmetric i.e. \( R(\xi, X).P = 0 \) holds, then the manifold is an Einstein manifold where \( P \) is projective curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

If a Lorentzian \( \beta \)-Kenmotsu manifold admits conformal Ricci soliton then using the same calculation we can obtain similar result, only the value of constant \( A \) will be changed which would not hamper our main result. Hence we can state the following theorem:

**Theorem 5.2** A Lorentzian \( \beta \)-Kenmotsu manifold admitting conformal Ricci soliton and is projectively semi symmetric i.e. \( R(\xi, X).P = 0 \) holds, then the manifold is an Einstein manifold where \( P \) is projective curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.
§6. Lorentzian $\beta$-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi, X).\tilde{P} = 0$

Let $M$ be a $n$ dimensional Lorentzian $\beta$-Kenmotsu manifold admitting Ricci soliton $(g, V, \lambda)$. The pseudo projective curvature tensor $\tilde{P}$ on $M$ is defined by

$$\tilde{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y]$$

$$-\frac{r}{n}\left[\frac{a}{n-1}\right] + b|g(Y, Z)X - g(X, Z)Y|. \quad (6.1)$$

Here we consider that the manifold is pseudo projectively semi symmetric i.e. $R(\xi, X).\tilde{P} = 0$ holds.

So

$$R(\xi, X)(\tilde{P}(Y, Z)W) - \tilde{P}(R(\xi, X)Y, Z)W - \tilde{P}(Y, R(\xi, X)Z)W - \tilde{P}(Y, Z)R(\xi, X)W = 0, \quad (6.2)$$

for all vector fields $X, Y, Z, W$ on $M$.

Using (2.7) and putting $W = \xi$ in (6.2) we have

$$\eta(\tilde{P}(Y, Z)\xi)X - g(X, \tilde{P}(Y, Z)\xi)\xi - \eta(Y)\tilde{P}(X, Z)\xi + g(X, Y)\tilde{P}(\xi, Z)\xi$$

$$-\eta(Z)\tilde{P}(Y, X)\xi + g(X, Z)\tilde{P}(Y, \xi)\xi - \eta(\xi)\tilde{P}(Y, Z)X + \eta(X)\tilde{P}(Y, Z)\xi = 0. \quad (6.3)$$

Now from (6.1) we can write

$$\tilde{P}(X, Y)\xi = aR(X, Y)\xi + b[S(Y, \xi)X - S(X, \xi)Y] + \frac{r}{n}\left[\frac{a}{n-1}\right] + b|g(Y, \xi)X - g(X, \xi)Y|. \quad (6.4)$$

Using (2.1), (2.8), (2.15) in (6.4) and after a long calculation we get

$$\tilde{P}(X, Y)\xi = \varphi(\eta(X)Y - \theta(Y)X), \quad (6.5)$$

where $\varphi = (a\beta^2 - Ab - \frac{r}{n}\left[\frac{a}{n-1}\right] + b]$.

Using (6.5) and putting $Z = \xi$ in (6.3) we obtain

$$\tilde{P}(Y, \xi)X + \varphi\eta(X)Y - \varphi g(X, Y)\xi = 0. \quad (6.6)$$

Taking inner product with $\xi$ in (6.6) we get

$$\eta(\tilde{P}(Y, \xi)X) + \varphi\eta(X)\eta(Y) - \varphi g(X, Y) = 0. \quad (6.7)$$

Again from (6.1) we can write

$$\tilde{P}(X, \xi)Z = a(X, \xi)Z + b[S(\xi, Z)X - S(X, Z)\xi] + \frac{r}{n}\left[\frac{a}{n-1}\right] + b|g(\xi, Z)X - g(X, Z)\xi|. \quad (6.8)$$
Using (2.1), (2.7), (2.15) in (6.8) we get

\[
P(X, \xi)Z = a\beta^2 [g(X, Z)\xi - \eta(Z)X] + b[A\eta(Z)X - S(X, Z)\xi] \\
+ \frac{r}{n} \left[ \frac{a}{n - 1} + b \right] g(\xi, Z)X - g(X, Z)\xi.
\]  

(6.9)

Taking inner product with \( \xi \) and replacing \( X \) by \( Y \), \( Z \) by \( X \) in (6.9) we have

\[
\eta(\tilde{P}(Y, \xi)X) = a\beta^2 [-g(X, Y) - \eta(X)\eta(Y)] + b[A\eta(X)\eta(Y) + S(X, Y)] + \\
\frac{r}{n} \left[ \frac{a}{n - 1} + b \right] [\eta(X)\eta(Y) - g(X, Y)].
\]  

(6.10)

Using (6.10) in (6.7) and after a brief simplification we obtain

\[
S(X, Y) = T g(X, Y) + U \eta(X)\eta(Y),
\]  

(6.11)

where \( T = -\frac{1}{6}[\alpha - \beta^2 - \frac{r}{n} \left[ \frac{a}{n - 1} + b \right] - \varphi] \) and \( U = -\frac{1}{6}[\varphi + \frac{r}{n} \left[ \frac{a}{n - 1} + b \right] + Ab - a\beta^2]. \)

From (6.11) we can conclude that the manifold is \( \eta \)-Einstein. Thus we have the following theorem:

**Theorem 6.1** If a Lorentzian \( \beta \)-Kenmotsu manifold admits Ricci soliton and is pseudo projectively semi symmetric i.e. \( R(\xi, X) \hat{P} = 0 \) holds, then the manifold is \( \eta \) Einstein manifold where \( \hat{P} \) is pseudo projective curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

If a Lorentzian \( \beta \)-Kenmotsu manifold admits conformal Ricci soliton then by following the same calculation we would obtain the same result, only the constant value of \( T \) and \( U \) will be changed. Hence we can state the following theorem:

**Theorem 6.2** A Lorentzian \( \beta \)-Kenmotsu manifold admitting conformal Ricci soliton and is pseudo projectively semi symmetric i.e. \( R(\xi, X) \hat{P} = 0 \) holds, then the manifold is \( \eta \) Einstein manifold where \( \hat{P} \) is pseudo projective curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

§7. An Example of a 3-Dimensional Lorentzian \( \beta \)-Kenmotsu Manifold

In this section we construct an example of a 3-dimensional Lorentzian \( \beta \)-Kenmotsu manifold. To construct this, we consider the three dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\} \) where \( (x, y, z) \) are the standard coordinates in \( \mathbb{R}^3 \). The vector fields

\[
e_1 = e^{-z} \frac{\partial}{\partial x}, e_2 = e^{-z} \frac{\partial}{\partial y}, e_3 = e^{-z} \frac{\partial}{\partial z}
\]

are linearly independent at each point of \( M \).
Let $g$ be the Lorentzian metric defined by

\[ g(e_1, e_1) = 1, \ g(e_2, e_2) = 1, \ g(e_3, e_3) = -1, \]
\[ g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0. \]

Let $\eta$ be the 1-form which satisfies the relation

\[ \eta(e_3) = -1. \]

Let $\phi$ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then we have

\[ \phi^2(Z) = Z + \eta(Z)e_3, \]
\[ g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W) \]

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. Now, after calculating we have

\[ [e_1, e_3] = e^{-z}e_1, \ [e_1, e_2] = 0, \ [e_2, e_3] = e^{-z}e_2. \]

The Riemannian connection $\nabla$ of the metric is given by the Koszul's formula which is

\[ 2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y) \]
\[ -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \] (7.1)

By Koszul’s formula we get

\[ \nabla_{e_1}e_1 = e^{-z}e_3, \ \nabla_{e_2}e_1 = 0, \ \nabla_{e_3}e_1 = 0, \]
\[ \nabla_{e_1}e_2 = 0, \ \nabla_{e_2}e_2 = 'e^{-z}e_3, \ \nabla_{e_3}e_2 = 0, \]
\[ \nabla_{e_1}e_3 = e^{-z}e_1, \ \nabla_{e_2}e_3 = e^{-z}e_2, \ \nabla_{e_3}e_3 = 0. \]

From the above we have found that $\beta = e^{-z}$ and it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a Lorentzian $\beta$-kenmotsu manifold. The results established in this note can be verified on this manifold.

References


