Some Properties of Conformal \(\beta\)-Change

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Abstract: We have considered the conformal \(\beta\)-change of the Finsler metric given by

\[ L(x,y) \rightarrow \bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y)), \]

where \(\sigma(x)\) is a function of \(x\), \(\beta(x,y) = b_i(x)y^i\) is a 1-form on the underlying manifold \(M^n\), and \(f(L(x,y), \beta(x,y))\) is a homogeneous function of degree one in \(L\) and \(\beta\). We have studied quasi-C-reducibility, C-reducibility and semi-C-reducibility of the Finsler space with this metric. We have also calculated V-curvature tensor and T-tensor of the space with this changed metric in terms of v-curvature tensor and T-tensor respectively of the space with the original metric.

Key Words: Conformal change, \(\beta\)-change, Finsler space, quasi-C-reducibility, C-reducibility, semi-C-reducibility, V-curvature tensor, T-tensor.


§1. Introduction

Let \(F^n = (M^n, L)\) be an \(n\)-dimensional Finsler space on the differentialble manifold \(M^n\) equipped with the fundamental function \(L(x,y)\). B.N.Prasad and Bindu Kumari and C. Shibata [1,2] have studied the general case of \(\beta\)-change, that is, \(L^*(x,y) = f(L, \beta)\), where \(f\) is positively homogeneous function of degree one in \(L\) and \(\beta\). They have also calculated the relationships between some important tensors of \((M^n, L)\) and the corresponding tensors of \((M^n, \bar{L})\), but have also studied several properties of this change.

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We have changed the order of combination of the above two changes in our paper [6], where we have applied $\beta$-change first and conformal change afterwards, i.e.,
\[ \tilde{L}(x, y) = e^{\sigma(x)} f(L(x, y), \beta(x, y)), \] (1.1)
where $\sigma(x)$ is a function of $x$, $\beta(x, y) = b_i(x)y^i$ is a 1-form. We have called this change as conformal $\beta$-change of Finsler metric. In this paper we have investigated the condition under which a conformal $\beta$-change of Finsler metric leads a Douglas space into a Douglas space. We have also found the necessary and sufficient conditions for this change to be a projective change.

In the present paper, we investigate some properties of conformal $\beta$-change. The Finsler space equipped with the metric $\tilde{L}$ given by (1.1) will be denoted by $\tilde{F}^n$. Throughout the paper the quantities corresponding to $\tilde{F}^n$ will be denoted by putting bar on the top of them. We shall denote the partial derivatives with respect to $x^i$ and $y^i$ by $\partial_i$ and $\hat{\partial}_i$ respectively. The Fundamental quantities of $F^n$ are given by
\[ g_{ij} = \hat{\partial}_i \hat{\partial}_j \frac{L^2}{2} = h_{ij} + l_i l_j, \quad l_i = \hat{\partial}_i L. \]

Homogeneity of $f$ gives
\[ Lf_1 + \beta f_2 = f, \] (1.2)
where subscripts 1 and 2 denote the partial derivatives with respect to $L$ and $\beta$ respectively. Differentiating above equations with respect to $L$ and $\beta$ respectively, we get
\[ Lf_{12} + \beta f_{22} = 0 \quad \text{and} \quad Lf_{11} + \beta f_{21} = 0. \] (1.3)
Hence we have
\[ f_{11}/\beta^2 = (-f_{12})/L\beta = f_{22}/L^2, \] (1.4)
which gives
\[ f_{11} = \beta^2 \omega, \quad f_{12} = -L\beta \omega, \quad f_{22} = L^2 \omega, \] (1.5)
where Weierstrass function $\omega$ is positively homogeneous of degree -3 in $L$ and $\beta$. Therefore
\[ L\omega_1 + \beta \omega_2 + 3\omega = 0, \] (1.6)
where $\omega_1$ and $\omega_2$ are positively homogeneous of degree -4 in $L$ and $\beta$. Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that $f$ is not linear function of $L$ and $\beta$ so that $\omega \neq 0$.

The concept of concurrent vector field has been given by Matsumoto and K. Eguchi [11] and S. Tachibana [17], which is defined as follows:

The vector field $b_i$ is said to be a concurrent vector field if
\[ b_{ij} = -g_{ij} \quad \text{and} \quad b_i|_j = 0, \] (1.7)
where small and long solidus denote the h- and v-covariant derivatives respectively. It has been
proved by Matsumoto that \( b_i \) and its contravariant components \( b^i \) are functions of coordinates alone. Therefore from the second equation of (1.7), we have \( C_{ijk}b^i = 0 \).

The aim of this paper is to study some special Finsler spaces arising from conformal \( \beta \)-change of Finsler metric, viz., quasi-C-reducible, C-reducible and semi-C-reducible Finsler spaces. Further, we shall obtain v-curvature tensor and T-tensor of this space and connect them with v-curvature tensor and T-tensor respectively of the original space.

§2. Metric Tensor and Angular Metric Tensor of \( \bar{F}^n \)

Differentiating equation (1.1) with respect to \( y^i \) we have

\[
\bar{l}_i = e^\sigma (f_1 l_i + f_2 b_i). \tag{2.1}
\]

Differentiating (2.1) with respect to \( y^j \), we get

\[
\bar{h}_{ij} = e^{2\sigma} \left( \frac{ff_1}{L} h_{ij} + f L^2 \omega m_i m_j \right), \tag{2.2}
\]

where \( m_i = b_i - \beta \frac{L}{L_l} l_i \).

From (2.1) and (2.2) we get the following relation between metric tensors of \( F^n \) and \( \bar{F}^n \):

\[
\bar{g}_{ij} = e^{-2\sigma} \left[ \frac{L}{ff_1} g^{ij} - \frac{p\beta}{L} l_i l_j + (f L^2 \omega + f_2^2) b_i b_j + p(b_i l_j + b_j l_i) \right], \tag{2.3}
\]

where \( p = f_1 f_2 - \beta L \omega \).

The contravariant components \( \bar{g}^{ij} \) of the metric tensor of \( \bar{F}^n \), obtainable from \( \bar{g}^{ij} \bar{g}_{jk} = \delta^i_k \), are as follows:

\[
\bar{g}^{ij} = e^{-2\sigma} \left[ \frac{L}{ff_1} g^{ij} + \frac{p L^3}{f^3 f_1 t} \left( \frac{f \beta}{L^2} - \Delta f_2 \right) t^i t^j - \frac{L^4 \omega}{f^2 f_1 t} b^i b^j - \frac{p L^2}{f^2 f_1 t} (l^i b^j + l^j b^i) \right], \tag{2.4}
\]

where \( t^i = g^{ij} l_j \), \( b^2 = b_i b^i \), \( b^i = g^{ij} b_j \), \( g^{ij} \) is the reciprocal tensor of \( g_{ij} \) of \( F^n \), and

\[
t = f_1 + L^3 \omega \Delta, \Delta = b^2 - \frac{\beta^2}{L^2}. \tag{2.5}
\]

\[
\begin{align*}
(a) \hat{s}_i f &= e^{\sigma} \left( \frac{f}{L} l_i + f_2 m_i \right), & (b) \hat{s}_i f_1 &= -e^{\sigma} \beta L \omega m_i, \\
(c) \hat{s}_i f_2 &= e^{\sigma} L^2 \omega m_i, & (d) \hat{s}_i p &= -\beta q L m_i, \\
(e) \hat{s}_i \omega &= -\frac{3\omega}{L} l_i + \omega_2 m_i, & (f) \hat{s}_i b^2 &= -2C_{..i}, \\
(g) \hat{s}_i \Delta &= -2C_{..i} - \frac{2\beta}{L^2} m_i.
\end{align*} \tag{2.6}
\]
(a) $\dot{q}_i = -\frac{3q}{L}l_i$,  
(b) $\dot{t} = -2L\dot{\omega}C_{..i} + L^3\dot{\omega}_2 - 3\beta L\dot{\omega}m_i$,  
(c) $\dot{t} = -\frac{3q}{L}l_i + (4f_2\omega_2 + 3\omega^2L^2 + f\omega_{22})m_i$. \hspace{1cm} (2.7)

§3. Cartan’s C-Tensor and C-Vectors of $\bar{F}^n$

Cartan’s covariant C-tensor $C_{ijk}$ of $F^n$ is defined by

$$\bar{C}_{ijk} = \frac{1}{4} \partial_i \partial_j \partial_k L^2 = \delta_k g_{ij}$$

and Cartan’s C-vectors are defined as follows:

$$C_i = C_{ijk}g^{jk}, C^i = C^i_{jk}g^{jk}. \hspace{1cm} (3.1)$$

We shall write $C^2 = C^i C_i$. Under the conformal $\beta$-change (1.1) we get the following relation between Cartan’s C-tensors of $F^n$ and $\bar{F}^n$:

$$\bar{C}_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L} C_{ijk} + \frac{p}{2L}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{qL^2}{2} m_i m_j m_k \right]. \hspace{1cm} (3.2)$$

We have

(a) $m_i l^i = 0$, 
(b) $m_i b^i = b^2 - \frac{\beta^2}{L^2} = \Delta = b_i m^i$, 
(c) $g_{ij} m^i = h_{ij} m^i = m_j$. \hspace{1cm} (3.3)

From (2.1), (2.2), (2.3) and (3.2), we get

$$C^h_{ij} = C^h_{ij} + \frac{p}{2f_1}(h_{ij} m^h + h_{ij} m_i + h_{ij} m_j) + \frac{qL^3}{2f_1} m_j m_k m^h$$

$$- \frac{L}{f} C_{ijk} n^h - \frac{pL}{2f^2 f_1} h_{jk} n^h - \frac{2pL + qL^2}{2f^2 f_1} m_j m_k n^h, \hspace{1cm} (3.4)$$

where $n^h = fL^2\omega^h + p^h$ and $h^h_i = g^h h_{ij}, C_{ij} = C_{rij} b^r, C_{..i} = C_{rji} b^r b^j$ and so on.

Proposition 3.1 Let $\bar{F}^n = (M^n, \tilde{L})$ be an $n$-dimensional Finsler space obtained from the conformal $\beta$-change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element $\tilde{l}_i$, angular metric tensor $\tilde{h}_{ij}$, fundamental metric tensor $\tilde{g}_{ij}$ and (h)hv-torsion tensor $\tilde{C}_{ijk}$ of $\bar{F}^n$ are given by (2.1), (2.2), (2.3) and (3.2), respectively.

From (2.4), (3.1), (3.2) and (3.4) we get the following relations between the C-vectors of of $F^n$ and $\bar{F}^n$ and their magnitudes

$$\bar{C}_i = C_i - L^3\omega C_{..i} + \mu m_i, \hspace{1cm} (3.5)$$
where
\[
\mu = \frac{p(n + 1)}{2ff_1} - \frac{3pL^3\omega\Delta}{2ff_1} + \frac{qL^3\Delta(1 - L^3\omega\Delta)}{2ff_1};
\]
\[
\bar{C}^i = \frac{e^{-2\sigma}L}{ff_1}C^i + M^i,
\]
(3.6)
where
\[
M^i = \frac{\mu e^{-2\sigma}L}{ff_1}m^i - \frac{L^4\omega}{ff_1}C^i - (C_i - e^{2\sigma}L^3\omega C_i + \mu \Delta) \left( \frac{L^3\omega}{ff_1}b^i + \frac{L}{ff}y^i \right)
\]
and
\[
\bar{C}^2 = \frac{e^{-2\sigma}}{p}C^2 + \lambda,
\]
(3.7)
where
\[
\lambda = \left( \frac{e^{-2\sigma}L}{ff_1} - L^3\omega\Delta \right) \mu^2 \Delta + \frac{2\mu e^{-2\sigma}L}{ff_1}C,
\]
\[
- (1 + 2\mu \Delta) \frac{L^3\omega}{ff_1} + (1 - 3\mu + e^{2\sigma}L^2\omega ff_1 C) L^3\omega C + L^3\omega C + \left( \frac{e^{2\sigma}L^2\omega}{ff_1}C_+ - 2C \right).
\]

§4. Special Cases of \(\bar{F}^n\)

In this section, following Matsumoto [10], we shall investigate special cases of \(\bar{F}^n\) which is conformally \(\beta\)-changed Finsler space obtained from \(F^n\).

**Definition 4.1** A Finsler space \((M^n, L)\) with dimension \(n \geq 3\) is said to be quasi-C-reducible if the Cartan tensor \(C_{ijk}\) satisfies
\[
C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j,
\]
(4.1)
where \(Q_{ij}\) is a symmetric indicatory tensor.

The equation (3.2) can be put as
\[
\bar{C}_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L}C_{ijk} + \frac{1}{6\pi(ijk)} \left\{ \left( \frac{3p}{L}h_{ij} + qL^2m_im_j \right) m_k \right\} \right],
\]
where \(\pi(ijk)\) represents cyclic permutation and sum over the indices \(i, j\) and \(k\).

Putting the value of \(m_k\) from equation (3.5) in the above equation, we get
\[
\bar{C}_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L}C_{ijk} + \frac{1}{6\mu}\pi(ijk) \left\{ \left( \frac{3p}{L}h_{ij} + qL^2m_im_j \right)(\bar{C}_k - C_k + L^3\omega C_+) \right\} \right].
\]
Rearranging this equation, we get
\[
\bar{C}_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L} C_{ijk} + \frac{1}{6\mu} \pi_{(ijk)} \left\{ \left( \frac{3p}{L} h_{ij} + qL^2 m_i m_j \right) \bar{C}_k \right\} + \frac{1}{6\mu} \pi_{(ijk)} \left\{ \left( \frac{3p}{L} h_{ij} + qL^2 m_i m_j \right) \left( L^3 \omega C_{k..} - C_k \right) \right\} \right].
\]

Further rearrangement of this equations gives
\[
\bar{C}_{ijk} = \pi_{(ijk)}(\bar{H}_{ij} \bar{C}_k) + U_{ijk}, \tag{4.2}
\]
where \(\bar{H}_{ij} = e^{2\sigma} \frac{ff_1}{6\mu} \left( \frac{3p}{L} h_{ij} + qL^2 m_i m_j \right)\), and
\[
U_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L} C_{ijk} + \frac{1}{6\mu} \pi_{(ijk)} \left\{ \left( \frac{3p}{L} h_{ij} + qL^2 m_i m_j \right) \left( L^3 \omega C_{k..} - C_k \right) \right\} \right]. \tag{4.3}
\]

Since \(\bar{H}_{ij}\) is a symmetric and indicatory tensor, therefore from equation (4.2) we have the following theorem.

**Theorem 4.1** Conformally \(\beta\)-changed Finsler space \(\bar{F}^n\) is quasi-C-reducible iff the tensor \(U_{ijk}\) of equation (4.3) vanishes identically.

We obtain a generalized form of Matsumoto’s result [10] as a corollary of the above theorem.

**Corollary 4.1** If \(F^n\) is Riemannian space, then the conformally \(\beta\)-changed Finsler space \(\bar{F}^n\) is always a quasi-C-reducible Finsler space.

**Definition 4.2** A Finsler space \((M^n, L)\) of dimension \(n \geq 3\) is called C-reducible if the Cartan tensor \(C_{ijk}\) is written in the form
\[
C_{ijk} = \frac{1}{n+1} (h_{ij} C_k + h_{ki} C_j + h_{jk} C_i). \tag{4.4}
\]

Define the tensor \(G_{ijk} = C_{ijk} - \frac{1}{n+1} (h_{ij} C_k + h_{ki} C_j + h_{jk} C_i)\). It is clear that \(G_{ijk}\) is symmetric and indicatory. Moreover, \(G_{ijk}\) vanishes iff \(F^n\) is C-reducible.

**Proposition 4.1** Under the conformal \(\beta\)-change(1.1), the tensor \(\bar{G}_{ijk}\) associated with the space \(\bar{F}^n\) has the form
\[
\bar{G}_{ijk} = e^{2\sigma} \frac{ff_1}{L} G_{ijk} + V_{ijk}, \tag{4.5}
\]
where
\[
V_{ijk} = \frac{1}{(n+1)} \pi_{(ijk)} \{ \left( e^{2\sigma} (n+1) (\alpha_1 h_{ij} + \alpha_2 m_i m_j) m_k + e^{2\sigma} \omega L^2 m_i m_j C_k \right) + e^{2\sigma} \omega L^2 (\bar{f} f_1 L \omega m_i m_j) C_{k..} \}, \tag{4.6}
\]
\[
\alpha_1 = \frac{e^{2\sigma} p}{2L} - \frac{\mu f f_1 e^{2\sigma}}{L(n+1)}, \quad \alpha_2 = \frac{e^{2\sigma} q L^2}{6} - \frac{\mu e^{2\sigma} \omega L^2}{(n+1)}.
\]
From (4.5) we have the following theorem.

**Theorem 4.2** Conformally $\beta$-changed Finsler space $\bar{F}^n$ is $C$-reducible iff $F^n$ is $C$-reducible and the tensor $V_{ijk}$ given by (4.6) vanishes identically.

**Definition 4.3** A Finsler space $(M^n, L)$ of dimension $n \geq 3$ is called semi-$C$-reducible if the Cartan tensor $C_{ijk}$ is expressible in the form:

$$C_{ijk} = \frac{r}{n+1}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) + \frac{s}{C^2}C_iC_jC_k,$$

where $r$ and $s$ are scalar functions such that $r + s = 1$.

Using equations (2.2), (3.5) and (3.7) in equation (3.2), we have

$$\bar{C}_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L}C_{ijk} + \frac{p}{2\mu ff_1}(h_{ij}\tilde{C}_k + h_{ki}\tilde{C}_j + \tilde{h}_{jk}\tilde{C}_i) + \frac{\Delta L(f_1q - 3p\omega)}{2f_1\mu C^2}\tilde{C}_i\tilde{C}_j\tilde{C}_k \right].$$

If we put

$$r' = \frac{p(n+1)}{2\mu ff_1}, s' = \frac{\Delta L(f_1q - 3p\omega)}{2f_1\mu t},$$

we find that $r' + s' = 1$ and

$$\bar{C}_{ijk} = e^{2\sigma} \left[ \frac{ff_1}{L}C_{ijk} + \frac{r'}{n+1}(h_{ij}\tilde{C}_k + h_{ki}\tilde{C}_j + \tilde{h}_{jk}\tilde{C}_i) + \frac{s'}{C^2}\tilde{C}_i\tilde{C}_j\tilde{C}_k \right].$$

From equation (4.8) we infer that $\bar{F}^n$ is semi-$C$-reducible iff $C_{ijk} = 0$, i.e. iff $F^n$ is a Riemannian space. Thus we have the following theorem.

**Theorem 4.3** Conformally $\beta$-changed Finsler space $\bar{F}^n$ is semi-$C$-reducible iff $F^n$ is a Riemannian space.

§5. v-Curvature Tensor of $\bar{F}^n$

The v-curvature tensor [10] of Finsler space with fundamental function $L$ is given by

$$S_{hijk} = C_{ijr}C^r_{hk} - C_{ikr}C^r_{hj}$$

Therefore the v-curvature tensor of conformally $\beta$-changed Finsler space $\bar{F}^n$ will be given by

$$\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj},$$

From equations (3.2) and (3.4), we have

$$\bar{C}_{ijr}\bar{C}^r_{hk} = e^{2\sigma} \left[ \frac{ff_1}{L}C_{ijr}C^r_{hk} + \frac{p}{2L}(C_{ijr}m_h + C_{ijh}m_k + C_{ihkm_j} + C_{hjk}m_i) + \frac{pf_1}{2L}(C_{ijr}h_hk + C_{hjk}h_{ijh}) - \frac{ff_1L^2}{t}\omega C_{ijr}C_{hk} \right].$$
The derivatives of $m_i$ where $\beta$ is a concurrent vector field, then $C_{ij} = 0$. Therefore the value of v-curvature tensor of $\tilde{F}^n$ as given by (5.3) is reduced to the extent that $d_{ij} = Rm_i m_j - Q h_{ij}$.

§6. The T-Tensor $T_{hijk}$

The T-tensor of $F^n$ is defined in [3] by

$$T_{hijk} = LC_{hij} \mid_k + C_{hij} l_k + C_{hik} l_j + C_{hjk} l_i + C_{ijk} l_h,$$

where

$$C_{hij} \mid_k = \hat{\partial}_k C_{hij} - C_{rij} C_{hk}^r - C_{hrj} C_{ik}^r - C_{hir} C_{jk}^r.$$

In this section we compute the T-tensor of $F^n$, which is given by

$$\tilde{T}_{hijk} = \tilde{L}C_{hij} \tilde{\mid}_k + \tilde{C}_{hij} \tilde{l}_k + \tilde{C}_{hik} \tilde{l}_j + \tilde{C}_{hjk} \tilde{l}_i + \tilde{C}_{ijk} \tilde{l}_h,$$

where

$$\tilde{C}_{hij} \tilde{\mid}_k = \hat{\partial}_k \tilde{C}_{hij} - \tilde{C}_{rij} \tilde{C}_{hk}^r - \tilde{C}_{hrj} \tilde{C}_{ik}^r - \tilde{C}_{hir} \tilde{C}_{jk}^r.$$

The derivatives of $m_i$ and $h_{ij}$ with respect to $y^k$ are given by

$$\hat{\partial}_k m_i = -\frac{\beta}{L^2} h_{ik} - \frac{1}{L} (l_i m_k), \quad \hat{\partial}_k h_{ij} = 2C_{ijk} - \frac{1}{L} (l_i h_{jk} + l_j h_{ki})$$
From equations (3.2) and (6.5), we have

\[
\hat{\partial}_k \tilde{C}_{hij} = e^{2\sigma} \left[ \frac{ff_1}{L} \partial_k C_{hij} + \frac{p}{L} (C_{ijk} m_h + C_{ijh} m_k + C_{ihk} m_j + C_{hjk} m_i) \\
- \frac{\beta}{2L^2} (h_{ij} h_{jk} + h_{ij} h_{ik} + h_{ih} h_{jk} + \frac{p}{2L^2} (h_{jk} h_{im} + h_{ik} h_{jm} + h_{ij} h_{lm}) \\
+ h_{ik} l_{jm} + h_{ik} l_{mj} + h_{jk} l_{im} + h_{jk} l_{jm}) - \frac{\beta q}{2} (h_{ij} h_{km}) \\
+ h_{ik} m_{im} + h_{ik} m_{jm} + h_{ik} m_{mk} + h_{ik} m_{mj} + h_{ik} m_{nj} + h_{ik} m_{jk} + h_{ik} m_{km} \\
- \frac{qL}{2} (h_{lm} m_{mk} + h_{lm} m_{mk} + l_{hm} m_{jm} + h_{mk} m_{mj} + h_{mk} m_{jm} + h_{mk} m_{mj} + h_{mk} m_{mj}) \\
+ \frac{L^2}{2} (4f_2 \omega_2 + 3L^2 \omega^2 + f \omega_2) m_h m_{mjm} \right]. \\
(6.6)
\]

Using equations (6.5) and (5.2) in equation (6.4), we get

\[
\tilde{C}_{hij} = e^{2\sigma} \frac{ff_1}{L} C_{hij |k} - e^{2\sigma} \frac{p}{2L} (C_{ijk} m_h + C_{ijh} m_k + C_{ihk} m_j + C_{hjk} m_i) \\
- \frac{\beta}{2L^2} (h_{ij} h_{jk} + h_{ij} h_{ik} + h_{ih} h_{jk}) - e^{2\sigma} \left( \frac{\beta q}{2} \right) \\
+ \frac{p^2 f_1 + pq f_1 L^3 \Delta + 3p^2}{4Lf f_1 t} (h_{ij} m_{mh} + h_{hkm} m_{mj} + h_{hm} m_{mk} + h_{hkm} m_{mj} + h_{hkm} m_{mj} \\
+ h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} \\
+ h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} \\
+ h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} \\
+ h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} + h_{hkm} m_{mj} - \frac{e^{2\sigma} qL}{2} (l_j m_{mh} m_k \\
+ l_j m_{mh} m_k + l_j m_{mh} m_k + l_j m_{mh} m_k) - \frac{p f_1 e^{2\sigma}}{2Lt} (C_{ij} h_{hk}) \\
+ C_{hj} h_{ik} + C_{jk} h_{ij} + C_{ik} h_{jk} + C_{ih} h_{jk} + C_{hj} h_{ik} + e^{2\sigma} \frac{f f_1 L^2 \omega}{t} (C_{ij} C_{hh}) \\
+ C_{hj} C_{ik} + C_{ik} C_{jk} - \frac{e^{2\sigma} \int L^2 (q f_1 - 2p \omega)}{2t} (C_{ij} m_{mh}) \\
+ C_{hj} m_{mj} + C_{ij} m_{mk} + C_{jk} m_{mj} \\
+ C_{hj} m_{mj} + C_{jkm} m_{mj} + e^{2\sigma} \frac{L^2 (4f_2 \omega_2 + 3L^2 \omega^2 + f \omega_2)}{2} \\
- \frac{3L^2 (2pq + (q f_1 - 2p \omega)(2p + L^3 \Delta))}{4f f_1 t} m_{mjm} m_{mk}. \\
(6.7)
\]

Using equations (2.1), (3.2) and (6.6) in equation (6.3), we get the following relation
between T-tensors of Finsler spaces $F^n$ and $\tilde{F}^n$:

$$
\tilde{T}_{hij} = e^{3\sigma} \left[ \frac{f^2 f_1}{L^2} T_{hij} + \frac{f(f_1 f_2 + f \beta \omega \lambda)}{2L} (C_{ijk} m_h + C_{ijh} m_k + C_{ihk} m_j) + C_{hijk} m_i \right] + \frac{f^2 f_1 L^2 \omega}{t} (C_{ij} C_{h,j} + C_{h,j} C_{,i} + C_{,h} C_{,jk}) - \frac{pf_1}{2L} (C_{ij} h_{hk} + C_{h,j} h_{ik} + C_{,h} h_{ij} + C_{,hk} h_{jh} + C_{,hk} h_{ji}) - \frac{fL^2(qf_1 - 2p \omega)}{2t} (C_{ij} m_k m_h + C_{,hk} m_i m_j + C_{,hk} m_i j) + C_{,ik} m_j m_h + C_{,ih} m_j m_k + C_{,jk} m_i m_j - \frac{p(2f \beta t + L^2 p \Delta)}{4L^3 t} (h_{ij} h_{hk}) - \left( \frac{p^2 f_1}{4L f_1} + \frac{pq f_1 L^3 \Delta + 3p^2 t + \beta q f}{2} - \frac{pf_2}{L} \right) (h_{ij} m_k m_j + h_{hk} m_i m_j + h_{dj} m_j m_k + h_{ik} m_i m_j) + \frac{L^2(4f_{2\omega} + 3L^2 \omega^2 + f\omega_{22})}{2} + 2L^2 f_2 q - \frac{3L^2(2pqf - (q f_1 - 2p \omega)(2p + L^3 q \Delta)}{4f_1 t} \right] m_i m_j m_k m_k. \tag{6.8}
$$

**Proposition 6.1** The relation between T-tensors of $F^n$ and $\tilde{F}^n$ is given by (6.7).

If $bi$ is a concurrent vector field in $F^n$, then $C_{ij} = 0$. Therefore from (6.8), we have

$$
\tilde{T}_{hij} = e^{3\sigma} \left[ \frac{f^2 f_1}{L^2} T_{hij} - \frac{p(2f \beta t + L^2 p \Delta)}{4L^3 t} (h_{ij} h_{hk} + h_{ij} h_{ik} + h_{ik} h_{jk}) - \left( \frac{p^2 f_1}{4L f_1} + \frac{pq f_1 L^3 \Delta + 3p^2 t + \beta q f}{2} - \frac{pf_2}{L} \right) (h_{ij} m_i m_j + h_{hk} m_i m_j + h_{dj} m_j m_k + h_{ik} m_i m_j) + \frac{L^2(4f_{2\omega} + 3L^2 \omega^2 + f\omega_{22})}{2} + 3L^2(qf_1 - 2p \omega)(2p + L^3 q \Delta)}{4L f_1 t} \right] m_i m_j m_k m_k. \tag{6.9}
$$

If $bi$ is a concurrent vector field in $F^n$, with vanishing T-tensor then T-tensor of $F^n$ is given by

$$
\tilde{T}_{hij} = e^{3\sigma} \left[ - \frac{p(2f \beta t + L^2 p \Delta)}{4L^3 t} (h_{ij} h_{hk} + h_{ij} h_{ik} + h_{ik} h_{jk}) - \left( \frac{p^2 f_1}{4L f_1} + \frac{pq f_1 L^3 \Delta + 3p^2 t + \beta q f}{2} - \frac{pf_2}{L} \right) (h_{ij} m_i m_j + h_{hk} m_i m_j + h_{dj} m_j m_k + h_{ik} m_i m_j) + \frac{L^2(4f_{2\omega} + 3L^2 \omega^2 + f\omega_{22})}{2} - \frac{3L^2(2pqf_1 - (q f_1 - 2p \omega)(2p + L^3 q \Delta)}{4L f_1 t} \right] m_i m_j m_k m_k. \tag{6.10}
$$
Some Properties of Conformal $\beta$-Change

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References


