The Dirac Hamiltonian's Egregious Violations of Special Relativity; the Nonrelativistic Pauli Hamiltonian's Unique Relativistic Extension

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Abstract

A single-particle Hamiltonian independent of the particle's coordinate ensures the particle conserves momentum, i.e., is free. If the Hamiltonian's energy-momentum is also Lorentz-covariant, it is uniquely determined by the particle's rest energy, and the particle has speed below c and constant velocity parallel to its conserved momentum (Newton's First Law), so its orbital angular momentum is conserved. Dirac set the square of his free-particle Hamiltonian equal to the square of this Hamiltonian, but he unwittingly ruined his Hamiltonian's energy-momentum Lorentz-covariance by making it inhomogeneously linear in momentum. A Dirac particle's speed is thus independent of its momentum, so by elimination can involve only c. Dirac free-particle speed comes out fixed to c times the square root of three, and the same fixed speed is obtained with the electromagnetically minimally coupled Dirac Hamiltonian, destroying the very basis of the textbook idea that Dirac Hamiltonians reduce to nonrelativistic Pauli Hamiltonians for weak fields and nonrelativistic particle speed. Dirac Hamiltonians egregiously violate the special-relativistic speed limit c, so must be discarded, and the actual relativistic extension of the nonrelativistic Pauli Hamiltonian worked out, which is done via Lorentz-invariant upgrade of the rest-frame Pauli action functional. The relativistic Pauli Hamiltonian is obtained in closed form for zero external magnetic field, otherwise a successive approximation scheme applies.

Introduction

The *relativistic Hamiltonian* $H(\mathbf{p})$ *for a free particle* ensures *conservation of the particle's momentum* $\mathbf{p}$ through *its independence of the particle's coordinate* $\mathbf{r}$. The *Lorentz covariance of the Hamiltonian's associated energy-momentum four-vector* $H^\mu = (H(\mathbf{p}), c\mathbf{p})$ allows it to be worked out for an arbitrary value of $\mathbf{p}$ *from its value at* $\mathbf{p} = \mathbf{0}$ *where* $H^\mu = (H_0, \mathbf{0})$*, and* $H_0 = H(\mathbf{p} = \mathbf{0})$ *is the particle's rest energy*. We now Lorentz transform $(H_0, \mathbf{0})$ *from* an inertial frame where the free particle has zero momentum *to* an inertial frame where it has some arbitrary velocity $\dot{\mathbf{r}}$ such that $|\dot{\mathbf{r}}| < c$. Zero momentum *corresponds to zero velocity because as* $|\mathbf{p}| \to 0$*, we have the familiar nonrelativistic relation of velocity to momentum*, i.e., $\dot{\mathbf{r}} = (\mathbf{p}/m)$. The Lorentz transformation of $(H_0, \mathbf{0})$ *from* the inertial frame where the particle has zero velocity, $\dot{\mathbf{r}} = \mathbf{0}$, *to* the inertial frame where the particle has an arbitrary velocity $\dot{\mathbf{r}}$ such that $|\dot{\mathbf{r}}| < c$ is,

$$(H_0, \mathbf{0}) \to H_0 \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}} (1, (\dot{\mathbf{r}}/c)) = H^\mu = (H(\mathbf{p}), c\mathbf{p}). \tag{1a}$$

We read off from Eq. (1a) that,

$$\mathbf{p} = (H_0/c)(\dot{\mathbf{r}}/c) \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}, \tag{1b}$$

which is readily inverted to obtain $(\dot{\mathbf{r}}/c)$,

$$(\dot{\mathbf{r}}/c) = (c\mathbf{p}/H_0) \left(1 + |c\mathbf{p}/H_0|^2\right)^{-\frac{1}{2}}, \tag{1c}$$

which permits us to in addition obtain,

$$\left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}} = \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}. \tag{1d}$$

We now insert Eqs. (1d) and (1c) into Eq. (1a) in order to obtain H^μ in terms of $\mathbf{p}$ instead of in terms of $\dot{\mathbf{r}}$,

$$H^\mu = \left(H_0 \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}},\, c\mathbf{p}\right) = (H(\mathbf{p}),\, c\mathbf{p}), \tag{1e}$$

which yields the relativistic free-particle Hamiltonian $H(\mathbf{p})$,

$$H(\mathbf{p}) = H_0 \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}, \tag{1f}$$

where, as pointed out in the paragraph preceding Eq. (1a), $H_0 = H(\mathbf{p} = \mathbf{0})$ is the particle's *rest energy*. The $|\mathbf{p}| \to 0$ asymptotic form $(H(\mathbf{p}) - H_0) \sim \left(|c\mathbf{p}|^2/(2H_0)\right)$ is obviously *equal to the particle's nonrelativistic kinetic energy* $\left(|\mathbf{p}|^2/(2m)\right)$, so $H_0 = mc^2$, which implies that $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$.

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The *central idea* which guided Dirac's 1928 development of his purportedly "relativistic" free-particle Hamiltonian operator $H_D(\mathbf{p})$ was his *intuitive impression* that the resulting *free-particle* Schrödinger equation,

$$i\hbar\partial\psi/\partial t = H_D(\mathbf{p})\psi, \tag{2a}$$

(whose Hamiltonian $H_D(\mathbf{p})$ is *independent* of $\mathbf{r}$ to render the particle's momentum *constant* in accord with the particle's being *free*), *must be space-time symmetric in configuration representation in order to accord with special relativity* [1]. Since in configuration representation, $\mathbf{p}\psi$ is given by,

$$\mathbf{p}\psi = -i\hbar\nabla_{\mathbf{r}}\psi, \tag{2b}$$

Dirac *specifically implemented* his somewhat vague *intuitive impression* that the Eq. (2a) free-particle Schrödinger equation is space-time symmetric by *postulating* that $H_D(\mathbf{p})$ *is inhomogeneously linear in* $\mathbf{p}$, namely that [1, 2, 3, 4],

$$H_D(\mathbf{p}) = c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2, \tag{2c}$$

where $\vec{\alpha}$ and β are, of course, Hermitian, dimensionless and *independent of* $\mathbf{p}$ *and* $\mathbf{r}$.

The Heisenberg equations of motion with this $H_D(\mathbf{p})$ then yield,

$$\dot{\mathbf{p}} = (-i/\hbar)[\mathbf{p}, H_D(\mathbf{p})] = (-i/\hbar)[\mathbf{p}, c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2] = \mathbf{0}, \tag{2d}$$

which of course is the basic property of a free particle, namely that its momentum is conserved, and they *also* yield [5, 6],

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H_D(\mathbf{p})] = (-i/\hbar)[\mathbf{r}, c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2] = c\vec{\alpha}, \tag{2e}$$

which, since $\vec{\alpha}$ is *independent of* $\mathbf{p}$, unfortunately outright *contradicts* one of the fundamental attributes of special relativity, namely that the $|\mathbf{p}| \to 0$ asymptotic form of its free-particle relationships *accord with the corresponding free-particle relationships of Newtonian physics*, which in *this* instance implies that,

$$\dot{\mathbf{r}} \sim (\mathbf{p}/m) \text{ as } |\mathbf{p}| \to 0. \tag{2f}$$

The *incompatibility with* Eq. (2f) of the Eq. (2e) consequence $\dot{\mathbf{r}} = c\vec{\alpha}$ of the Dirac free-particle Hamiltonian $H_D(\mathbf{p})$, where $\vec{\alpha}$ *is independent of* $\mathbf{p}$, shows that the Dirac free-particle Hamiltonian $H_D(\mathbf{p})$ of Eq. (2c) *violates special relativity*. This is *the first of an almost endless list of examples* of ways in which *the Dirac theory violates special relativity*. In due course, we shall stop paying any heed whatsoever to the Dirac theory, but for the time being there remain quite a few thought-provoking illustrative examples of its relativistic illegitimacy still to be broached.

We can determine that the Dirac free-particle *speed* $|\dot{\mathbf{r}}| = c|\vec{\alpha}|$ *is a fixed c-number* whose specific value *is under no circumstances compatible with special relativity*, upon noting the algebraic properties of $\vec{\alpha}$ and β which follow from Dirac's *second* postulate involving the *square* of the free-particle relativistic energy $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$ [1, 7, 8],

$$(H_D(\mathbf{p}))^2 = (H(\mathbf{p}))^2 = m^2c^4 + |c\mathbf{p}|^2, \tag{3a}$$

which ensures that any solution of the Dirac equation satisfies the Klein-Gordon equation. In conjunction with Eq. (2c), it *also* turns out to *ensure* that the Dirac equation *shares* the Klein-Gordon equation's property of having *negative-energy free-particle solutions*. The well-known algebraic consequences of Eq. (3a) for $\vec{\alpha}$ and β are [1, 7, 8],

$$(\alpha_x)^2 = (\alpha_y)^2 = (\alpha_z)^2 = (\beta)^2 = 1 \text{ and } \alpha_x,\ \alpha_y,\ \alpha_z \text{ and } \beta \text{ mutually anticommute.} \tag{3b}$$

Thus for the Dirac free-particle speed $|\dot{\mathbf{r}}|$, Eqs. (2e) and (3b) yield,

$$|\dot{\mathbf{r}}| = c|\vec{\alpha}| = c\left((\alpha_x)^2 + (\alpha_y)^2 + (\alpha_z)^2\right)^{\frac{1}{2}} = c(1+1+1)^{\frac{1}{2}} = c\sqrt{3}, \tag{3c}$$

a fixed c-number whose specific fixed value $c\sqrt{3}$ not only egregiously violates the nonrelativistic asymptotic free-particle requirement that $|\dot{\mathbf{r}}| \sim (|\mathbf{p}|/m)$ as $|\mathbf{p}| \to 0$, but which as well egregiously violates the special-relativistic free-particle *speed limit* $|\dot{\mathbf{r}}| < c$.

Although some textbooks *do* actually point out that *the eigenvalues of each of the three components* of the Dirac "free-particle" *velocity* $\dot{\mathbf{r}} = c\vec{\alpha}$ are $\pm c$ [5], which *immediately mathematically implies that* $|\dot{\mathbf{r}}| = c\sqrt{3}$, there is apparently *no* textbook *which dares to actually write down this result that* $|\dot{\mathbf{r}}| = c\sqrt{3}$, that reveals with utterly devastating succinctness the complete theoretical-physics worthlessnes of Dirac's 1928 "relativistic free-particle" Schrödinger-equation efforts.

The electromagnetically minimally coupled Dirac Hamiltonian [9, 10],

$$H_D(\mathbf{r}, \mathbf{P}) = \vec{\alpha} \cdot (c\mathbf{P} - e\mathbf{A}) + e\phi + \beta mc^2, \tag{4a}$$

is immediately seen to have the *same* velocity consequence [6],

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H_D(\mathbf{r}, \mathbf{P})] = (-i/\hbar)[\mathbf{r},\, c\vec{\alpha} \cdot \mathbf{P}] = c\vec{\alpha}, \tag{4b}$$

as the "free-particle" Dirac Hamiltonian, which implies that *any* electromagnetically coupled Dirac particle *always* has the speed $|\dot{\mathbf{r}}| = c\sqrt{3}$ that egregiously violates the special-relativistic particle speed limit $|\dot{\mathbf{r}}| < c$.

This *speed result*, $|\dot{\mathbf{r}}| = c\sqrt{3}$, for the electromagnetically minimally coupled Dirac Hamiltonian of Eq. (4a) *immediately gives the lie* to the well-known textbook "theorem" that that Hamiltonian *effectively reduces* to the electromagnetically coupled nonrelativistic Pauli Hamiltonian [11, 12],

$$H = \big(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\big) + e\phi - (e\hbar/(2mc))(\vec{\sigma} \cdot \mathbf{B}), \tag{5a}$$

in the latter's *region of special-relativistic validity*, which is, of course, when,

$$|\dot{\mathbf{r}}| = (|\mathbf{P} - (e/c)\mathbf{A}|/m) \ll c, \tag{5b}$$

because of the fact that,

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H] = (-i/\hbar)\left[\mathbf{r}, \big(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\big)\right] = ((\mathbf{P} - (e/c)\mathbf{A})/m). \tag{5c}$$

However, *there is no overlap whatsoever between* $|\dot{\mathbf{r}}| = c\sqrt{3}$ *and* $|\dot{\mathbf{r}}| \ll c$, so this well-known textbook "theorem" *comically falls flat on its face*.

The purported "proof" which textbooks proffer for this well-known "theorem" relies on the ostensibly "plausible" supposition for the Dirac Hamiltonian that if [13, 14],

$$|\mathbf{P} - (e/c)\mathbf{A}| \ll mc, \tag{6a}$$

then,

$$|E - mc^2| \ll mc^2. \tag{6b}$$

The *difficulty* with this "plausible" supposition becomes apparent when the Dirac equation's *unavoidable negative-energy solutions* are taken into consideration. For example, it is *entirely feasible* to have the condition given by Eq. (6a) *in coexistence with*,

$$E \approx -mc^2, \tag{6c}$$

which, of course, *drastically violates* the ostensibly "plausible" supposition of Eq. (6b).

The *insuperable underlying problems* with the Dirac Hamiltonian are Dirac's *two disastrously false ideas* that space-time symmetry of the Schrödinger equation *can supplant Lorentz covariance of the Hamiltonian operator's associated energy-momentum operator* and that *the utterly extraneous negative-energy solutions* of the Klein-Gordon equation *can be permitted to infiltrate the Schrödinger equation*.

Consider the generic single-particle Schrödinger equation in configuration representation,

$$i\hbar\partial\psi/\partial t = H(\mathbf{r}, \mathbf{P})\psi. \tag{7a}$$

If it *accords* with special relativity, its Hamiltonian operator $H(\mathbf{r}, \mathbf{P})$ *perforce* is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r}, \mathbf{P}), c\mathbf{P})$ *is Lorentz-covariant*. It *also turns out* that *if* this associated energy-momentum operator of the Hamiltonian operator of such a generic Schrödinger equation *is* Lorentz-covariant, then that Schrödinger equation *is the time component of a Lorentz-covariant four-vector equation system* whose *three space components* follow from *just* the familiar configuration-representation fact that,

$$\mathbf{P}\psi = -i\hbar\nabla_{\mathbf{r}}\psi. \tag{7b}$$

To demonstrate this, we *first* point out that the Eq. (7a) generic single-particle Schrödinger equation in configuration representation *together with* Eq. (7b) yields the four-equation system,

$$i\hbar c\,\partial\psi/\partial x_\mu = H^\mu\psi, \tag{7c}$$

which written out in detail is,

$$i\hbar(\partial\psi/\partial t, -c\nabla_{\mathbf{r}}\psi) = (H(\mathbf{r},\mathbf{P})\psi, c\,\mathbf{P}\psi). \tag{7d}$$

This four-equation system is satisfied because *its time component* is *precisely* the Eq. (7a) generic Schrödinger equation, and *its three space components* are equivalent to,

$$-i\hbar\nabla_{\mathbf{r}}\psi = \mathbf{P}\psi, \tag{7e}$$

which is *precisely* Eq. (7b).

In *addition* to *merely* the straightforward *validity* of the Eq. (7c) four-equation system, it is the case that *since* the space-time differential operator,

$$i\hbar c\,\partial/\partial x_\mu = i\hbar(\partial/\partial t, -c\nabla_{\mathbf{r}}),$$

manifestly is a Lorentz-covariant four-vector operator, *if* the Hamiltonian operator $H(\mathbf{r},\mathbf{P})$ of the Eq. (7a) generic Schrödinger equation is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r},\mathbf{P}), c\mathbf{P})$ *is as well* a Lorentz-covariant four-vector operator, then the Eq. (7c) four-equation system clearly *is a Lorentz-covariant four-vector equation system* whose *time component* of course is the Eq. (7a) generic Schrödinger equation, and whose three space components follow from *just* the familiar configuration-representation fact that Eq. (7b) holds. Therefore *if* the Hamiltonian operator $H(\mathbf{r},\mathbf{P})$ of the Eq. (7a) generic Schrödinger equation is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r},\mathbf{P}), c\mathbf{P})$ *is* a Lorentz-covariant four-vector operator, then the Eq. (7a) generic Schrödinger equation clearly *accords* with special relativity. The converse of this statement is self-evident, so *a necessary and sufficient condition* for the Eq. (7a) generic Schrödinger equation to *accord* with special relativity is that its Hamiltonian operator $H(\mathbf{r},\mathbf{P})$ is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r},\mathbf{P}), c\mathbf{P})$ is a Lorentz-covariant four-vector operator.

Also, since a generic single-particle, configuration-representation Schrödinger equation which *accords* with special relativity is *only the time component* of a Lorentz-covariant four-vector equation system, *it absolutely cannot be space-time symmetric*. Therefore, *making* such a generic Schrödinger equation *space-time symmetric* will *always* produce a *result* which *violates* some aspect of special relativity, which is the *opposite* of Dirac's *intuitive impression* [1, 2, 3, 4], and provides an explanation *why* the Dirac Hamiltonian violates special relativity.

We also note the arcane fact that since a generic single-particle, configuration-representation Schrödinger equation which *accords* with special relativity *is only the time component* of a Lorentz-covariant four-vector equation system, *no homogeneously-linear recasting* of such a generic Schrödinger equation that *accords* with special relativity *is Lorentz-transformation form-invariant*. This arcane fact is *only* of interest because the Dirac equation has been ad hoc *retrofitted* with a custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions under which the Dirac equation multiplied by the Dirac matrix β is *form-invariant* [15]. Since the Dirac equation's multiplication by the Dirac matrix β is *indeed* a homogeneously-linear recasting of the Dirac equation, the above arcane fact tells us that if this ad hoc retrofitted custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions *really is* the Lorentz transformation that it is *claimed* to be, then *the Dirac equation*, whose multiplication by the Dirac matrix β is *form-invariant* under this presumed Lorentz transformation, *definitely cannot be in accord* with special relativity. On the other hand, if this ad hoc retrofitted custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions *isn't really* the Lorentz transformation that it is *claimed* to be, then we must, of course, *directly examine* the Dirac equation to *check* whether it accords with special relativity. For example, the *free particle* Dirac Hamiltonian is inhomogeneously *linear* in $\mathbf{p}$, which we have seen in detail above *isn't* in accord with special relativity. The *moral* of this *overlong story* about the ad hoc retrofitted custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions, under which the Dirac equation's multiplication by the Dirac matrix β is *form-invariant* [15], is that *its existence in absolutely no way* demonstrates that the Dirac equation is in *accord* with special relativity; indeed, if this transformation of Dirac's four-component wave functions *really is* the Lorentz transformation which *it is claimed to be*, then the Dirac equation *definitely isn't in accord* with special relativity.

We now point out that the relativistic Hamiltonian operator $H(\mathbf{p})$ for a free particle that is given by Eq. (1f), which we reproduce here, namely,

$$H(\mathbf{p}) = H_0 \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}, \tag{1f}$$

is outright incompatible with Dirac's assertion that $H(\mathbf{p})$ is inhomogeneously *linear* in $\mathbf{p}$. Since the above-reproduced Eq. (1f) Hamiltonian operator $H(\mathbf{p})$ for a free particle follows from *just* the principles of special relativity (i.e., the *Lorentz-covariance* of the energy-momentum four-vector operator $H^\mu = (H(\mathbf{p}), c\mathbf{p})$), it is *clear* that the well-known Dirac "free-particle" Hamiltonian operator of Eq. (2c), namely,

$$H_D(\mathbf{p}) = c\vec{\alpha} \cdot \mathbf{p} + \beta mc^2, \tag{8a}$$

which is inhomogeneously *linear* in $\mathbf{p}$, *must contradict special relativity.*

Application of the Heisenberg equation of motion using the above-reproduced Eq. (1f) *actual* relativistic free-particle Hamiltonian operator $H(\mathbf{p}) = H_0 \left((1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}$ yields the following result for the *actual* relativistic free-particle velocity operator $\dot{\mathbf{r}}$,

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H(\mathbf{p})] = (-i/\hbar)[\mathbf{r}, H_0 \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}] = (c^2\mathbf{p}/H_0) \left(1 + |c\mathbf{p}/H_0|^2\right)^{-\frac{1}{2}}, \tag{8b}$$

which *already* was obtained in Eq. (1c) as a consequence of Lorentz-transformation considerations. The asymptotic form as $|\mathbf{p}| \to 0$ of the relativistic $\dot{\mathbf{r}}$ of Eq. (8b) is,

$$\dot{\mathbf{r}} \sim (c^2\mathbf{p}/H_0), \tag{8c}$$

which we set equal to the well-known *nonrelativistic velocity*,

$$\dot{\mathbf{r}} = (\mathbf{p}/m), \tag{8d}$$

to obtain $H_0 = mc^2$, which we in turn insert into Eq. (1f) to obtain the mass m relativistic free-particle Hamiltonian $H(\mathbf{p})$,

$$H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}. \tag{8e}$$

We likewise insert $H_0 = mc^2$ into Eq. (8b) to obtain the mass m *relativistic* free-particle velocity $\dot{\mathbf{r}}$,

$$\dot{\mathbf{r}} = (\mathbf{p}/m) \left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}}. \tag{8f}$$

Unlike the pathological results $\dot{\mathbf{r}} = c\vec{\alpha}$ of Eq. (2e) and $|\dot{\mathbf{r}}| = c\sqrt{3}$ of Eq. (3c) of the Dirac "free-particle" Hamiltonian $H_D(\mathbf{p}) = c\vec{\alpha} \cdot \mathbf{p} + \beta mc^2$ of Eq. (2c), the *correct* Eq. (8e) relativistic free-particle Hamiltonian $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$ yields the Eq. (8f) free-particle relativistic velocity $\dot{\mathbf{r}}$ that is *not* independent of $\mathbf{p}$ as the Dirac $c\vec{\alpha}$ is, that *is* parallel to $\mathbf{p}$ and therefore yields *conservation of orbital angular momentum*, which the Dirac $c\vec{\alpha}$ *fails to*, which *does* have the correct nonrelativistic $|\mathbf{p}| \to 0$ asymptotic behavior $\dot{\mathbf{r}} \sim (\mathbf{p}/m)$, but the Dirac $c\vec{\alpha}$ *utterly fails to*, which *does* satisfy the relativistic free-particle speed limit $|\dot{\mathbf{r}}| < c$, whereas the Dirac $c\vec{\alpha}$ *always* corresponds to the *fixed unphysical speed* $|\dot{\mathbf{r}}| = c\sqrt{3}$, and whose components *do* commute with each other, whereas those of the the Dirac $c\vec{\alpha}$ pathologically *anticommute* with each other.

Of course the *correct* Eq. (8e) relativistic free-particle Hamiltonian $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$ implies that a relativistic free particle *experiences no acceleration whatsoever*, in accord with Newton's First Law, as immediately follows from insertion of Eqs. (8f) and (8e) into the Heisenberg equation of motion,

$$\ddot{\mathbf{r}} = (-i/\hbar)[\dot{\mathbf{r}}, H(\mathbf{p})] = (-i/\hbar) \left[(\mathbf{p}/m) \left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}}, \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}\right] = \mathbf{0}. \tag{8g}$$

The *pathological* Dirac "free-particle" Hamiltonian $H_D(\mathbf{p}) = c\vec{\alpha} \cdot \mathbf{p} + \beta mc^2$ of Eq. (2c), however, puts forth an astoundingly *conflicting* tale about the spontaneous acceleration of free particles, namely,

$$\ddot{\mathbf{r}} = (-i/\hbar)[\dot{\mathbf{r}}, H_D(\mathbf{p})] = (-i/\hbar) \left[c\vec{\alpha}, c\vec{\alpha} \cdot \mathbf{p} + \beta mc^2\right] = \left(-ic^2/\hbar\right) ((\mathbf{p} \times (\vec{\alpha} \times \vec{\alpha})) + (2\vec{\alpha}\beta mc)), \tag{8h}$$

which implies that *when* $\mathbf{p} = \mathbf{0}$, namely in the case of a Dirac free particle of *zero momentum*,

$$\ddot{\mathbf{r}} = -2i\vec{\alpha}\beta \left(mc^3/\hbar\right), \tag{8i}$$

and therefore,

$$|\ddot{\mathbf{r}}| = 2\sqrt{3}\left(mc^3/\hbar\right), \tag{8j}$$

so a $\mathbf{p} = \mathbf{0}$ Dirac "free particle" undergoes spontaneous acceleration (due to varying *direction of travel* at *special-relativity violating fixed speed* $c\sqrt{3}$), completely *unlike any relativistically-legitimate free particle described by* Eq. (8g). When we put the mass of the electron into the Eq. (8j) spontaneous acceleration expression, we obtain an acceleration magnitude $|\ddot{\mathbf{r}}|$ of the *astounding* order of 10^{28} times g, where $g = 9.8$ meters per second squared, the acceleration of gravity at the Earth's surface. The *mind-boggling magnitude* of this putative spontaneous acceleration is traced to the occurrence of $\hbar$ in the *denominator* of Eq. (8j). The Dirac "free-particle" Hamiltonian's twisted "ability" to cause the commonplace physical phenomenon of acceleration *to come out inversely proportional to* $\hbar$ shows *how ill-considered a kluge the Dirac Hamiltonian is*. Dirac *began* by *defying elementary physics knowledge* about the momentum character of free-particle Hamiltonians, *both* Newtonian and relativistic, by bullheadedly decreeing *his* "free-particle" Hamiltonian $H_D(\mathbf{p})$ of Eq. (2c) to be *inhomogeneously linear in the free-particle momentum*, which he *followed* with *the clumsily naive afterthought of requiring* $(H_D(\mathbf{p}))^2$ *to equal* $(H(\mathbf{p}))^2$ *in order to paper over the consequent theoretical-physics damage*, that in fact *only serves to makes matters worse* by foisting *wildly physically-inappropriate anticommuting properties on the venerable concepts of velocity and rest energy*. A more idiosyncratically ham-handed, tin-ear performance than Dirac's can scarcely be imagined.

As a salient example *of the physically-muddled nature* of the Dirac theory due to the Dirac "velocity" operator $\dot{\mathbf{r}}$ having the *anticommuting* components $c\alpha_x$, $c\alpha_y$ and $c\alpha_z$ instead of physically sensible *commuting* components, the "famous" Dirac spin-1/2 operator $\mathbf{S}$, namely,

$$\mathbf{S} = -i(\hbar/4)(\vec{\alpha}\times\vec{\alpha}),$$

can, since $\dot{\mathbf{r}} = c\vec{\alpha}$, obviously just as well be written,

$$\mathbf{S} = -i\left(\hbar/\left(4c^2\right)\right)(\dot{\mathbf{r}}\times\dot{\mathbf{r}}),$$

so *the very existence* of the "famous" Dirac spin-1/2 operator $\mathbf{S}$ *is the entirely artificial consequence of Dirac-theory physically-pathological velocity-component anticommutation in the place of physically-sensible velocity-component commutation*.

Scrutiny of Eq. (8h) above, which exhibits *the underlying causes* of special-relativity violating *spontaneous acceleration* $\ddot{\mathbf{r}}$ *of a Dirac "free particle"* reveals that *existence* of the Dirac spin-1/2 operator-related entity $(\vec{\alpha}\times\vec{\alpha})$ *definitely contributes to the special-relativity violating spontaneous acceleration* $\ddot{\mathbf{r}}$ *of a Dirac "free particle"*.

The "automatic emergence" of the spin-1/2 operator $\mathbf{S} = -i(\hbar/4)(\vec{\alpha}\times\vec{\alpha}) = -i\left(\hbar/\left(4c^2\right)\right)(\dot{\mathbf{r}}\times\dot{\mathbf{r}})$ in Dirac theory is traditionally *touted* as "a great accomplishment" of that theory, but (1) *its very existence* depends on *the pathological anticommuting property of the components of the particle velocity* $\dot{\mathbf{r}}$ *in Dirac theory*, and (2) Eq. (8h) tells us that the spin-1/2 operator-related entity $(\vec{\alpha}\times\vec{\alpha})$ contributes to *the special-relativity violating spontaneous acceleration* $\ddot{\mathbf{r}}$ of a Dirac "free particle". The Dirac theory's spin-1/2 operator $\mathbf{S} = -i(\hbar/4)(\vec{\alpha}\times\vec{\alpha}) = -i\left(\hbar/\left(4c^2\right)\right)(\dot{\mathbf{r}}\times\dot{\mathbf{r}})$ is a completely *chimerical* consequence of *the egregiously unphysical velocity-component anticommutation feature* of that theory.

There is *no doubt* that the electromagnetically minimally coupled Dirac Hamiltonian of Eq. (4a),

$$H_D(\mathbf{r},\mathbf{P}) = \vec{\alpha}\cdot(c\mathbf{P} - e\mathbf{A}) + e\phi + \beta mc^2,$$

which egregiously violates special relativity because its particle velocity $\dot{\mathbf{r}}$ *equals* $c\vec{\alpha}$ *and therefore its particle speed* $|\dot{\mathbf{r}}| = c\sqrt{3}$ *always exceeds* c, cannot correctly describe single-particle relativistic quantum mechanics, and thus *definitely cannot be a physically sensible relativistic extension of the electromagnetically coupled nonrelativistic Pauli Hamiltonian of* Eq. (5a), namely,

$$H = \left(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\right) + e\phi - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}),$$

which apparently is physically unobjectionable in the nonrelativistic regime, namely when,

$$|\dot{\mathbf{r}}| = (|\mathbf{P} - (e/c)\mathbf{A}|/m) \ll c.$$

However, a powerful way to produce dynamical theories guaranteed to accord with special relativity is by utilizing *Lorentz-invariant action functionals*. Furthermore, the relativistic physics of a single particle is *identical* to its nonrelativistic physics *when that particle is at rest*. Thus *given a nonrelativistic single-particle dynamical theory which one wishes to upgrade to its relativistic counterpart*, one proceeds by working out *its nonrelativistic action functional*, and then specializes that nonrelativistic action functional to *zero particle velocity*, which entity *is the base that is to be upgraded to a Lorentz-invariant action functional* which is valid at any particle velocity whose magnitude is less than c.

Given a *nonrelativistic Hamiltonian* which is to be upgraded to its *relativistic counterpart*, a great many steps are necessary. One must pass from the nonrelativistic Hamiltonian to the corresponding nonrelativistic Lagrangian, thence to the nonrelativistic action functional, which is specialized to *zero* particle velocity. This is *the base to be upgraded to the Lorentz-invariant action functional, whose integrand then yields the relativistic Lagrangian*, from which one passes to the relativistic Hamiltonian. A *caveat* here is that passages between Lagrangians and Hamiltonians entail solution of algebraic equations, *which isn't always feasible in closed analytic form*.

Action-based unique relativistic extension of the Pauli Hamiltonian

In preparation for the relativistic extension of the nonrelativistic Pauli Hamiltonian of Eq. (5a), we add to it the particle's rest-mass energy mc^2,

$$H = mc^2 + \big(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\big) + e\phi - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}). \tag{9a}$$

Note that the addition of such a constant term to a Hamiltonian in no way changes the quantum Heisenberg or classical Hamiltonian equations of motion.

To obtain the nonrelativistic action S_{nr} which corresponds to the Hamiltonian H of Eq. (9a), we first work out the Lagrangian L which corresponds to that Hamiltonian H. The conversion of such a particle Hamiltonian to a particle Lagrangian requires swapping the Hamiltonian's dependence on the canonical three-momentum $\mathbf{P}$ for the Lagrangian's dependence on the particle's three-velocity $\dot{\mathbf{r}}$. We obtain that particle three-velocity $\dot{\mathbf{r}}$ from the Heisenberg equation of motion (or alternatively, in this case, from the equivalent classical Hamiltonian equation of motion),

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H] = \nabla_{\mathbf{P}} H = (\mathbf{P} - (e/c)\mathbf{A})/m. \tag{9b}$$

We now *invert* the relation of Eq. (9b) between particle velocity $\dot{\mathbf{r}}$ and canonical momentum $\mathbf{P}$ to read,

$$\mathbf{P} = m\dot{\mathbf{r}} + (e/c)\mathbf{A}, \tag{9c}$$

and insert it into the well-known relationship of the Lagrangian to the Hamiltonian, namely,

$$L = \dot{\mathbf{r}}\cdot\mathbf{P} - H\Big|_{\mathbf{P}=m\dot{\mathbf{r}}+(e/c)\mathbf{A}} = -mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}), \tag{9d}$$

from which we immediately obtain the nonrelativistic action,

$$S_{\mathrm{nr}} = \int L dt = \int \big[-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})\big]\, dt.$$

Of course we don't want the nonrelativistic action S_{nr} itself, but its *specialization* S to the case of *zero particle velocity*, namely $\dot{\mathbf{r}} = \mathbf{0}$,

$$S = \int \big[-mc^2 - e\phi + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})\big]\, dt. \tag{9e}$$

We shall undertake the Lorentz-invariant upgrade of the three terms of this action S individually. The first term of S which we tackle is that of the free particle,

$$S^0 = \int (-mc^2) dt. \tag{10a}$$

To make S^0 Lorentz-invariant, we only need to replace the time differential dt by the Lorentz-invariant proper time differential $d\tau$,

$$d\tau = \big((dt)^2 - |d\mathbf{r}/c|^2\big)^{\frac{1}{2}} = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}\, dt. \tag{10b}$$

Therefore,

$$d\tau/dt = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}}, \tag{10c}$$

and from this it of course follows that,

$$dt/d\tau = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}. \tag{10d}$$

The Lorentz-invariant upgraded S^0 therefore is,

$$S^0_{\mathrm{rel}} = \int (-mc^2) d\tau. \tag{10f}$$

Eq. (10f), by use of Eq. (10c) can of course also be expressed as,

$$S^0_{\mathrm{rel}} = \int (-mc^2) \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} dt. \tag{10g}$$

We next tackle the part of the action S which encompasses the interaction of the particle's charge e with the electromagnetic potential ϕ,

$$S^e = \int (-e\phi) dt. \tag{11a}$$

We carry out the Lorentz-invariant upgrade of S^e by replacing the time differential dt in Eq. (11a) by the Lorentz-invariant time differential $d\tau$, and upgrading the $\dot{\mathbf{r}} = \mathbf{0}$ static-limit potential energy $e\phi$ to a dynamic Lorentz-invariant function of $\dot{\mathbf{r}}$. To do so we first rewrite the static potential energy $e\phi$ as the faux Lorentz invariant,

$$e\phi = eU_\mu(\dot{\mathbf{r}} = \mathbf{0})A^\mu, \tag{11b}$$

that has the faux Lorentz-covariant constituent,

$$U_\mu(\dot{\mathbf{r}} = \mathbf{0}) = \delta^0_\mu. \tag{11c}$$

which is valid *only* in the particle's rest frame where the particle's velocity $\dot{\mathbf{r}} = \mathbf{0}$. To upgrade the static faux Lorentz-covariant $U_\mu(\dot{\mathbf{r}} = \mathbf{0})$ to a dynamic true Lorentz-covariant entity $U_\mu(\dot{\mathbf{r}})$, we Lorentz-boost it from the particle's rest frame to the inertial frame where the particle has velocity $\dot{\mathbf{r}}$,

$$U_\mu(\dot{\mathbf{r}}) = U_\alpha(\dot{\mathbf{r}} = \mathbf{0})\Lambda^\alpha_\mu(\dot{\mathbf{r}}) = \delta^0_\alpha \Lambda^\alpha_\mu(\dot{\mathbf{r}}) = \Lambda^0_\mu(\dot{\mathbf{r}}). \tag{11d}$$

Therefore *the dynamic Lorentz-invariant upgrade of the static potential energy* $e\phi$ is,

$$eU_\mu(\dot{\mathbf{r}})A^\mu = e\Lambda^0_\mu(\dot{\mathbf{r}})A^\mu = e\gamma(\dot{\mathbf{r}})\,(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}), \tag{11e}$$

where,

$$\gamma(\dot{\mathbf{r}}) = (1 - (|\dot{\mathbf{r}}|^2/c^2))^{-\frac{1}{2}} = dt/d\tau. \tag{11f}$$

Thus the Lorentz-invariant upgrade of,

$$S^e = \int (-e\phi) dt,$$

is,

$$S^e_{\mathrm{rel}} = \int (-eU_\mu(\dot{\mathbf{r}})A^\mu) d\tau = \int (-e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}))dt. \tag{11g}$$

Finally we tackle the part of the action S that encompasses the interaction of the particle's spin with the magnetic field,

$$S^{\vec{\sigma}} = \int (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})dt. \tag{12a}$$

Again we replace the differential dt by the Lorentz-invariant differential $d\tau$ and upgrade the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$, which is valid in the $\dot{\mathbf{r}} = \mathbf{0}$ particle rest frame, to a dynamic Lorentz-invariant function of $\dot{\mathbf{r}}$. Preliminary to the upgrading of the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$, we write it as,

$$-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}) = -(e\hbar/(2mc))(\vec{\sigma}\cdot(\nabla\times\mathbf{A})) = (e\hbar/(2mc))\epsilon_{ijk}\sigma^i(\partial^j A^k). \tag{12b}$$

This representation of the static potential energy can be rewritten as the faux Lorentz invariant,

$$(e\hbar/(2mc))\epsilon_{ijk}\sigma^i(\partial^j A^k) = (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})(\partial^\mu A^\nu), \tag{12c}$$

that has the faux Lorentz-covariant constituent,

$$\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0}) = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0, \\ \epsilon_{ijk}\sigma^i & \text{if } \mu = j \text{ and } \nu = k,\ j, k = 1, 2, 3, \end{cases} \tag{12d}$$

which is valid *only* in the particle's rest frame where the particle's velocity $\dot{\mathbf{r}} = \mathbf{0}$. Note that $\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})$ is *antisymmetric* under the interchange of its two indices μ and ν. To upgrade the static faux Lorentz-covariant $\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})$ to a dynamic true Lorentz-covariant entity $\sigma_{\mu\nu}(\dot{\mathbf{r}})$, we Lorentz-boost it from the particle's rest frame to the inertial frame where the particle has velocity $\dot{\mathbf{r}}$,

$$\sigma_{\mu\nu}(\dot{\mathbf{r}}) = \sigma_{\alpha\beta}(\dot{\mathbf{r}} = \mathbf{0})\Lambda^\alpha_\mu(\dot{\mathbf{r}})\Lambda^\beta_\nu(\dot{\mathbf{r}}) = \epsilon_{ijk}\sigma^i\Lambda^j_\mu(\dot{\mathbf{r}})\Lambda^k_\nu(\dot{\mathbf{r}}). \tag{12e}$$

It is apparent from Eq. (12e) that the Lorentz-covariant second-rank tensor $\sigma_{\mu\nu}(\dot{\mathbf{r}})$ is *also* antisymmetric under the interchange of its two indices μ and ν. From Eqs. (12b) through (12e) it is clear that *the dynamic Lorentz-invariant upgrade of the static potential energy* $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$ is,

$$\begin{gathered}(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu) = (e\hbar/(2mc))\epsilon_{ijk}\sigma^i\Lambda^j_\mu(\dot{\mathbf{r}})\Lambda^k_\nu(\dot{\mathbf{r}})(\partial^\mu A^\nu) = \\ (e\hbar/(2mc))\left(\vec{\sigma}\cdot[(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)]\right),\end{gathered} \tag{12f}$$

where,

$$(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)^j \stackrel{\text{def}}{=} \Lambda^j_\mu(\dot{\mathbf{r}})\partial^\mu \quad\text{and}\quad (\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)^k \stackrel{\text{def}}{=} \Lambda^k_\nu(\dot{\mathbf{r}})A^\nu. \tag{12g}$$

The space components of the Lorentz boost of the four-vector partial-derivative operator,

$$\partial^\mu = ((1/c)(\partial/\partial t), -\nabla),$$

from the rest frame of the particle to the inertial frame in which the particle has velocity $\dot{\mathbf{r}}$ are given by,

$$(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu) = -\nabla - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\nabla) - \gamma(\dot{\mathbf{r}})(\dot{\mathbf{r}}/c)(1/c)(\partial/\partial t), \tag{12h}$$

and the space components of the *same* Lorentz boost of the electromagnetic four-vector potential,

$$A^\mu = (\phi, \mathbf{A}),$$

are given by,

$$(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu) = \mathbf{A} + (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A}) - \gamma(\dot{\mathbf{r}})(\dot{\mathbf{r}}/c)\phi. \tag{12i}$$

Using Eqs. (12h) and (12i) one can, with tedious effort, verify that,

$$\begin{gathered}(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu) = -(\nabla\times\mathbf{A}) - \\ (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[\nabla\times(\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A})) + (\dot{\mathbf{r}}\cdot\nabla)(\dot{\mathbf{r}}\times\mathbf{A})\right] - \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times(\dot{\mathbf{A}}/c) - \nabla\times((\dot{\mathbf{r}}/c)\phi)\right] = \\ -(\nabla\times\mathbf{A}) - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[\dot{\mathbf{r}}\times[-\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}]\right] + \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times\left[-\nabla\phi - (\dot{\mathbf{A}}/c)\right]\right] = \\ -(\nabla\times\mathbf{A}) - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[\dot{\mathbf{r}}\times[-\dot{\mathbf{r}}\times(\nabla\times\mathbf{A})]\right] + \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times\left[-\nabla\phi - (\dot{\mathbf{A}}/c)\right]\right] = \\ -\mathbf{B} - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[|\dot{\mathbf{r}}|^2\mathbf{B} - \dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{B})\right] + \gamma(\dot{\mathbf{r}})((\dot{\mathbf{r}}/c)\times\mathbf{E}) = \\ -\gamma(\dot{\mathbf{r}})\mathbf{B} + (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{B}) - \gamma(\dot{\mathbf{r}})(\mathbf{E}\times(\dot{\mathbf{r}}/c)).\end{gathered} \tag{12j}$$

From Eqs. (12f) and (12j) one sees that the dynamic Lorentz-invariant upgrade of the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$ is,

$$\begin{gathered}(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu) = (e\hbar/(2mc))\left(\vec{\sigma}\cdot[(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)]\right) = \\ -(e\hbar/(2mc))\left[\gamma(\dot{\mathbf{r}})(\vec{\sigma}\cdot\mathbf{B}) - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}(\vec{\sigma}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{B}) + \gamma(\dot{\mathbf{r}})(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))\right],\end{gathered} \tag{12k}$$

and thus the Lorentz-invariant upgrade of the Eq. (12a) spin contribution to the action, namely,

$$S^{\vec{\sigma}} = \int (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})dt.$$

comes out to be,

$$\begin{gathered} S^{\vec{\sigma}}_{\text{rel}} = -\int (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^{\mu}A^{\nu})d\tau = \\ \int (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) - (1-(\gamma(\dot{\mathbf{r}}))^{-1})|\dot{\mathbf{r}}|^{-2}(\vec{\sigma}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))\right]dt = \\ \int (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) - (1+(\gamma(\dot{\mathbf{r}}))^{-1})^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\vec{\sigma}\times\mathbf{E})\cdot(\dot{\mathbf{r}}/c)\right]dt, \end{gathered} \tag{12l}$$

as we see by using Eq. (12k) and the fact that,

$$\gamma(\dot{\mathbf{r}}) = (1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} = dt/d\tau.$$

In the last step of Eq. (12l), we have furthermore interchanged the "dot" $\cdot$ with the "cross" $\times$ in the triple scalar product,

$$(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c))),$$

and have as well applied the identity,

$$(1-(\gamma(\dot{\mathbf{r}}))^{-1})|\dot{\mathbf{r}}|^{-2} = (1+(\gamma(\dot{\mathbf{r}}))^{-1})^{-1}c^{-2}.$$

We are now in a position to write down the Lorentz-invariant upgrade S_{rel} of the nonrelativistic Pauli action S of Eq. (9e),

$$\begin{gathered} S_{\text{rel}} = S^0_{\text{rel}} + S^e_{\text{rel}} + S^{\vec{\sigma}}_{\text{rel}} = \int \left[-mc^2 - eU_{\mu}(\dot{\mathbf{r}})A^{\mu} - (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^{\mu}A^{\nu})\right]d\tau = \\ \int\Big[-mc^2(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + \\ (e\hbar/(2mc))\Big((\vec{\sigma}\cdot\mathbf{B}) - \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\Big)\Big]dt \end{gathered} \tag{13a}$$

From this Lorentz-invariant upgrade of the nonrelativistic Pauli action we can immediately write down the relativistic Pauli Lagrangian,

$$\begin{gathered} L_{\text{rel}} = -mc^2(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + \\ (e\hbar/(2mc))\Big((\vec{\sigma}\cdot\mathbf{B}) - \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\Big), \end{gathered} \tag{13b}$$

where, of course,

$$\mathbf{B} = \nabla\times\mathbf{A} \text{ and } \mathbf{E} = -\nabla\phi - (\dot{\mathbf{A}}/c). \tag{13c}$$

From Eq. (13b) we calculate *the relativistic Pauli Lagrangian's corresponding canonical momentum*,

$$\begin{gathered} \mathbf{P} = \nabla_{\dot{\mathbf{r}}}L_{\text{rel}} = m\dot{\mathbf{r}}(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + (e/c)\mathbf{A} + (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc^2))\left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}\Big[\vec{\sigma}((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))\mathbf{B} + \\ \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\dot{\mathbf{r}}/c)(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\Big]. \end{gathered} \tag{13d}$$

The *last three terms of* Eq. (13d), which all arise from *the relativistic distortion of the magnetic field* **B**, unfortunately *preclude solving analytically* for the particle's *velocity* $\dot{\mathbf{r}}$ in terms of the system's *canonical momentum* **P**. For that reason we cannot in general *analytically* parlay the relativistic Pauli system's *energy* E_{rel}, namely,

$$E_{\text{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\text{rel}}, \tag{13e}$$

into its relativistic Pauli *Hamiltonian* $H_{\text{rel}}(\mathbf{r}, \vec{\sigma}, \mathbf{P}, t)$. However we see from Eq. (13d) that the three offending terms which arise from the relativistic distortion of the magnetic field $\mathbf{B}$ are all *higher-order corrections in powers of* $|\dot{\mathbf{r}}/c|$, so we can easily rewrite Eq. (13d) as *a successive-approximation scheme* for the desired inversion of the canonical momentum $\mathbf{P}$ that is *consonant* with *the systematic carrying out of relativistic corrections*. The scheme *is considerably more transparent*, however, after all occurrences of the particle velocity $\dot{\mathbf{r}}$ on the right-hand side of Eq. (13d) (and as well on the right-hand side of Eq. (13e)) are *replaced* by occurrences *of the free-particle momentum* $\mathbf{p}$, which is,

$$\mathbf{p} \stackrel{\text{def}}{=} m\dot{\mathbf{r}}(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}, \text{ and implies,}$$
$$(\dot{\mathbf{r}}/c)(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} = \mathbf{p}/(mc), \quad (\dot{\mathbf{r}}/c) = \mathbf{p}(m^2c^2 + |\mathbf{p}|^2)^{-\frac{1}{2}}, \quad (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} = mc(m^2c^2 + |\mathbf{p}|^2)^{-\frac{1}{2}}. \tag{13f}$$

Using Eq. (13f) to eliminate all occurrences of the particle velocity $\dot{\mathbf{r}}$ on the right-hand side of Eq. (13d) in favor of the free-particle momentum $\mathbf{p}$ yields,

$$\begin{gathered}\mathbf{P} = \mathbf{p} + (e/c)\mathbf{A} + (e\hbar/(2mc^2))(\vec{\sigma} \times \mathbf{E}) - \\ (e\hbar/(2mc^2))\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1} \times \\ \left[\vec{\sigma}(\mathbf{p}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})\mathbf{B} + \left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(\mathbf{p}/(mc))(\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\right].\end{gathered} \tag{13g}$$

Eq. (13g) can now be readily rewritten *as a successive approximation scheme for the resolution of the free-particle momentum* $\mathbf{p}$ *in terms of the canonical momentum* $\mathbf{P}$,

$$\begin{gathered}\mathbf{p} = \mathbf{P} - (e/c)\mathbf{A} - (e\hbar/(2mc^2))(\vec{\sigma} \times \mathbf{E}) + \\ (e\hbar/(2mc^2))\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1} \times \\ \left[\vec{\sigma}(\mathbf{p}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})\mathbf{B} + \left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(\mathbf{p}/(mc))(\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\right].\end{gathered} \tag{13h}$$

In order for these successive approximations to $\mathbf{p}$ in terms of $\mathbf{P}$ to be able to produce successive approximations to the relativistic Pauli *Hamiltonian* H_{rel}, we must *also* banish all occurrences of the particle velocity $\dot{\mathbf{r}}$ in the system's *energy* E_{rel}, which is given on the right-hand side of Eq. (13e), in favor of the free-particle momentum $\mathbf{p}$.

We shall, however, *first* calculate that relativistic Pauli energy $E_{\text{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\text{rel}}$ of Eq. (13e) *entirely in terms of* $\dot{\mathbf{r}}$ by using the L_{rel} which is given by Eq. (13b) and the $\mathbf{P}$ which is given by Eq. (13d), and *then* use the relations given in Eq. (13f) to eliminate $\dot{\mathbf{r}}$ from E_{rel} in favor of $\mathbf{p}$.

From Eq. (13b) we obtain that,

$$\begin{gathered}-L_{\text{rel}} = mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} + e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) - \\ (e\hbar/(2mc))\Big((\vec{\sigma}\cdot\mathbf{B}) - \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\Big),\end{gathered} \tag{13i}$$

and from Eq. (13d) we obtain that,

$$\begin{gathered}\dot{\mathbf{r}}\cdot\mathbf{P} = m|\dot{\mathbf{r}}|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e(\dot{\mathbf{r}}/c)\cdot\mathbf{A} + (e\hbar/(2mc))(\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc))\left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) \times \\ \left[2 + \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}|\dot{\mathbf{r}}/c|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\right]\end{gathered} \tag{13j}$$

The complicated structure of the last term of Eq. (13j) can be simplified markedly, with the result,

$$\begin{gathered}\dot{\mathbf{r}}\cdot\mathbf{P} = m|\dot{\mathbf{r}}|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e(\dot{\mathbf{r}}/c)\cdot\mathbf{A} + (e\hbar/(2mc))(\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc))(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\end{gathered} \tag{13k}$$

Putting Eqs. (13i) and (13k) together produces,

$$E_{\text{rel}} = \dot{\mathbf{r}} \cdot \mathbf{P} - L_{\text{rel}} = mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e\phi -$$
$$(e\hbar/(2mc))\Big[(\vec{\sigma} \cdot \mathbf{B}) + (\vec{\sigma} \cdot (\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c) \cdot \mathbf{B}) \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1} (1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\Big]. \tag{13l}$$

We now use the relations given by Eq. (13f) to express the E_{rel} of Eq. (13l) entirely in terms of free-particle momentum $\mathbf{p}$ instead of in terms of the particle velocity $\dot{\mathbf{r}}$,

$$E_{\text{rel}} = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}} + e\phi -$$
$$(e\hbar/(2mc))\Big[(\vec{\sigma} \cdot \mathbf{B}) + (\vec{\sigma} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{B}) \left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1} (mc)^{-1}\Big]. \tag{13m}$$

Eq. (13m) is to be used in conjunction with the Eq. (13h) successive approximation scheme for obtaining the free-particle momentum $\mathbf{p}$ in terms of the canonical momentum $\mathbf{P}$, in order to generate successive approximations to the relativistic Pauli Hamiltonian H_{rel}.

In those cases where $\mathbf{B} = \mathbf{0}$, Eq. (13h) immediately yields the *exact* relationship of the canonical momentum $\mathbf{P}$ to the free-particle momentum $\mathbf{p}$, namely,

$$\mathbf{p} = \mathbf{P} - (e/c)\mathbf{A} - (e\hbar/(2mc^2))(\vec{\sigma} \times \mathbf{E}), \tag{13n}$$

and for *those* $\mathbf{B} = \mathbf{0}$ cases Eq. (13m) yields the following the *exact* relativistic Pauli Hamiltonian, namely,

$$H_{\text{rel}} = \left(m^2c^4 + |c\mathbf{P} - e\mathbf{A} - (e\hbar/(2mc))(\vec{\sigma} \times \mathbf{E})|^2\right)^{\frac{1}{2}} + e\phi. \tag{13o}$$

The relativistically extended Pauli Hamiltonian of Eq. (13o) clearly bears *a very close resemblance* to the *relativistic Lorentz Hamiltonian*, which describes a *spinless* relativistic charged particle interacting with an electromagnetic field. That notwithstanding, the *relativistically extended* Pauli Hamiltonian of Eq. (13o) *also* very clearly *incorporates* the interaction of a *moving* particle's *spin* with *an electric field*, a phenomenon *that is utterly and completely foreign to the the nonrelativistic Pauli Hamiltonian* of Eq. (1), which Eq. (13o) *exactly relativistically extends in those special cases where* $\mathbf{B} = \mathbf{0}$. The *purely relativistic* interaction of a *moving* particle's *spin* with *an electric field* is, of course *the essence* of the *hydrogen atom's spin-orbit interaction*. Thus the Eq. (13o) $\mathbf{B} = \mathbf{0}$ special case of the relativistically extended Pauli Hamiltonian is obviously useful for the hydrogen atom.

The very close resemblance to the physically irreproachable Lorentz Hamiltonian which the Eq. (13o) $\mathbf{B} = \mathbf{0}$ special case of the relativistically extended Pauli Hamiltonian manifests shows that the latter has *none* of the pathologies which are so typical of the Dirac Hamiltonian.

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