Constraining the Standard Model in Motivic Quantum Gravity

M D Sheppeard

4/32 Wyuna Ave, Freshwater, NSW 2096, Australia

E-mail: `marnisheppeard@gmail.com`

Abstract. A physical approach to a category of motives must account for the emergent nature of spacetime, where real and complex numbers play a secondary role to discrete operations in quantum computation. In quantum logic, the cardinality of a set is initially replaced by a dimension of a linear space, making contact with the increasing dimensions in an operad. The operad of associahedra governs tree level scattering, and is closely related to the permutohedra and cube tiles, where cube vertices can encode components of a spinor in higher dimensional octonionic approaches. A study of rest mass generation begins with the cosmological infrared scale, set by the neutrino masses, and its related see-saw mechanism. We employ the anyonic ribbon spectrum for Standard Model states, and consider its relation to magic star algebras, giving a context for the Koide rest mass phenomenology of charged leptons and quarks.

1. Overview

What is motivic quantum gravity? Algebraic geometers study periods associated to Feynman integrals in particle physics. Periods form a ring $\mathbb{P} \subset \mathbb{C}$, and we might construct this ring from smaller rings in the plane. As physicists, we seek canonical discrete structures for quantum gravity, rich enough to underlie a universal cohomology theory within which all physical amplitudes are computed. In particular, the lattice of $\mathbf{e_8}$ is embedded in $\mathbb{Z}^8/2$, which in turn is mapped densely into $\mathbb{C}$ [1] using only the integers of the golden ring extension of $\mathbb{Q}$, where $\phi = (1+\sqrt{5})/2$ and $\sigma = \sqrt{\phi+2}$ define real numbers of the form

$$a + b\phi + c\sigma + d\phi\sigma, \tag{1}$$

for integral a, b, c, d. The lattice defines the Penrose pentagonal tiling.

Thus our philosophy is *not* to begin with ordinary varieties or manifolds, but to consider the difficult question of their emergence from combinatorial data associated to quantum gravitational logic. Then any real or complex space, including spacetime or the full algebra of $\mathbf{e_8}$, become secondary to the algorithms that generate them. Since quantum mechanics naturally defines a symmetric monoidal category [2][3], one expects that its extension to gravity will also employ higher dimensional category theory in a fundamental way, combining generalized associahedra with canonical algebraic data.

Physically, the hypothesis of non local right handed neutrino states is capable of solving many cosmological conundrums. After the 2010 discovery [4][5][6] of the exact correspondence between a neutrino rest mass and the present day temperature of the CMB, it was natural to consider a new IR scale defined by the overall neutrino scale at around 0.01 eV, as considered in the neutrino condensate picture [7]. Successful predictions included a computation of the observable mass of our universe, tighter constraints on neutrino masses and an effective sterile mass for oscillation anomalies [8][9]. Connections to holography were discussed in [6]. The crucial mass photon relation comes from Wien's law

$$mc^2 = \beta T, \tag{2}$$

now justified by quantum inertia [10][11][12][13]. Neutrino states fit into the triplet ribbon spectrum of [14][15] for the Standard Model. Brannen showed in [16] how to extend Koide's formula [17][18] for the low energy charged lepton masses to neutrino masses. An inverted see-saw relation for the neutrino scale $m_\nu = 0.01$ eV,

$$m_H = \sqrt{m_\nu m_P} \tag{3}$$

suggestively approximates the Higgs mass.

Neutrino states are also expected to define the vacuum in the octonionic algebras of [19][20][21], as outlined in the next section. Section 3 introduces operad polytopes and a few important categorical concepts. According to Vaughan Jones, to understand the Standard Model, you need to see three monads. A proper monad is an endofunctor that generalizes the idempotent rule $P^2 = P$. Points are generically idempotents, either as objects in a Heyting algebra of open sets or as matrices in a Jordan algebra. A crucial monad in classical mathematics is the power set monad, and its simplest quantum analogue is determined by a symmetric monoidal category of vector spaces [22].

The importance of discretizing spaces was understood a long time ago by Nikola Tesla [23], who famously said that the Universe comes down to the magic of 3, 6, 9. These numbers show up as follows. Take the Fibonacci sequence, associated to powers of ϕ. Take the decimal parity of each number in the sequence. For example, the parity of 13 is $1 + 3 = 4$. As with binary codes, this defines a parity check bit. Now the entire Fibonacci sequence is a repeating list of 24 numbers,

$$1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9, \tag{4}$$

containing two copies of 3, 6 and 9, which label the vertices of a hexagon in Metatron's cube. Decimal parity is analogous to the check bit in a classical binary code. We expect that quantum gravity employs not only qubits, but also qutrits and 10-dits, to make up the base 60 of the universal clock. Tesla's code shows how the qutrit component separates from other numbers.

2. A topological particle spectrum

In the scheme of [14], a chiral set of massless Standard Model states is given as three strand ribbon diagrams. If we forget the dyonic braiding for the moment, states are still distinguished [15] by augmenting the underlying permutations in S_3. For this single generation, ν and $\overline{\nu}$ annihilate to a photon identity

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

We approach electroweak symmetry breaking in reverse, allowing mass to emerge from some abstract entanglement network. If the basis neutrinos are

$$\nu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \overline{\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6}$$

the electromagnetic charge is a set of twists on the three ribbon strands, where distinct twists may be assigned to each strand. We represent this by replacing 1*s* by one of three phases: 1 for neutral, ω for $+1/3$, or $\overline{\omega}$ for $-1/3$. Then the charged leptons are

$$e_L^- = \overline{\omega} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_R^+ = \omega \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7}$$

which indeed compose to the identity. Similarly,

$$e_L^+ = \omega \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_R^- = \overline{\omega} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

For quarks, put the charges onto individual strands, as in the colored matrices

$$u_L(1) = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}, \ u_L(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega & 0 & 0 \end{pmatrix}, \ u_L(3) = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix} \tag{9}$$

for left handed up quarks. Leptons are circulants while quarks are not. The $W^{\pm}$ bosons are given by

$$W^- = \overline{\omega} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W^+ = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

The tricky particle in this massless model is the Z boson, built from six remaining neutral boson matrices,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \quad \begin{pmatrix} \overline{\omega} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

and their three conjugates. These matrices permit a natural difermionic supersymmetry [15] through the *twisted Fourier* transform $\mathbf{F}$, defined on e_L^- by

$$\mathbf{F}(e_L^-) \equiv \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\omega} & \omega \\ 1 & \omega & \overline{\omega} \end{pmatrix} = W^-. \tag{12}$$

For right handed states we replace the mixed diagonal by its complex conjugate,

$$\mathbf{F}(e_R^-) \equiv \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \begin{pmatrix} 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \\ 0 & \overline{\omega} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\omega} & \omega \\ 1 & \omega & \overline{\omega} \end{pmatrix} = W^-. \tag{13}$$

In this way, circulant leptons map to electroweak bosons

$$e^{\pm} \mapsto W^{\pm}, \qquad \nu, \overline{\nu} \mapsto \gamma. \tag{14}$$

Since quarks are not circulant, the corresponding bosons are mixed phase non diagonals, presumably associated to the confinement of color.

The braid group B_3 extends the underlying permutations of S_3 and covers the modular group. In Fibonacci anyon categories, B_3 models $SU(2)$ in the construction of gates for universality in quantum computation [24]. It is also well known that the string net condensation of [25], using modular tensor categories, can recover QED and QCD with a Anderson-Higgs mechanism.

Recently, the relation of the ribbon spectrum to octonion algebras was clarified in [26], based on the work of Furey [21]. We start with the 64 dimensions of the $\mathbb{C}\otimes\mathbb{O}$ ideal algebra [19][20][21], which assigns $U(1)_Q$ and $SU(3)_C$ color charges to the quarks and leptons of the Standard Model. Selecting one octonion unit for the lepton doublet, define the $\mathbb{C}\otimes\mathbb{O}$ idempotent $\nu = (1+ie_7)/2$. The other six units $e_1, \cdots, e_6$ define

$$\alpha_1 = \frac{1}{2}(-e_5 + ie_4), \quad \alpha_2 = \frac{1}{2}(-e_3 + ie_1), \quad \alpha_3 = \frac{1}{2}(-e_6 + ie_2). \tag{15}$$

The Lie algebra generators Λ_a for $SU(3)_C$ occur in three different ways. Taking $(I, \nu, \overline{\nu})$ in $\mathbb{C}\otimes\mathbb{O}$, with $I = 1$,

$$\begin{aligned} \frac{1}{4}[\Lambda_a, \Lambda_b] &= \frac{i}{2} f_{abc}\Lambda_c \\ \frac{1}{4}[\Lambda_a\nu, \Lambda_b\nu] &= \frac{i}{2} f_{abc}\Lambda_c\nu \\ \frac{1}{4}[-\overline{\Lambda}_a\overline{\nu}, -\overline{\Lambda}_b\overline{\nu}] &= -\frac{i}{2} f_{abc}\overline{\Lambda}_c\overline{\nu}. \end{aligned} \tag{16}$$

Complex conjugation $i \mapsto -i$ sends particles to antiparticles. Charges for one generation [21] come from the number operator

$$N = \sum_{j=1}^{3} \alpha_j^\dagger \alpha_j, \tag{17}$$

with values in $\{0, 1, 2, 3\}$. Writing out the α_j components of N, a set of eight charges on a three qubit parity cube gives the set $\{\nu, d(3), \overline{u}(3), e^-\}$ from

$$A^\dagger A, \;\; \alpha_j A^\dagger A, \;\; \alpha_j\alpha_k A^\dagger A, \;\; \alpha_j\alpha_k\alpha_l A^\dagger A, \tag{18}$$

where

$$A \equiv \alpha_1\alpha_2\alpha_3 = i\nu. \tag{19}$$

The three copies of 64 for the generations suggest a (massless form) of triality for $\mathbf{e_8}$. Moving in this dirction, in (16) I represents LL^{-1}, where L and $R = L^{-1}$ are chiralities for the massless neutrino ν. Since ribbon diagrams will characterise chirality, we start with a basis for the Hopf algebra $\mathbb{C}C_3$,

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{20}$$

In every prime power dimension, the Fourier transform F_{p^r} diagonalizes 1-circulants, and there exists $p^r + 1$ mutually unbiased bases [27][28][29] generalizing the 2×2 Pauli matrices, providing a canonical matrix representation for multiplication in finite fields.

Now let ω be the primitive cubed root of unity. The 3×3 Fourier transform

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \tag{21}$$

diagonalizes 1-circulants, as in

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = F_3 \begin{pmatrix} r & \theta & \overline{\theta} \\ \overline{\theta} & r & \theta \\ \theta & \overline{\theta} & r \end{pmatrix} {F_3}^\dagger. \tag{22}$$

The circulant idempotents are

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \overline{\omega} \\ \overline{\omega} & 1 & \omega \\ \omega & \overline{\omega} & 1 \end{pmatrix}, \quad \overline{H} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \overline{\omega} & \omega \\ \omega & 1 & \overline{\omega} \\ \overline{\omega} & \omega & 1 \end{pmatrix}. \tag{23}$$

As Hermitian matrices, they belong to the Jordan algebra $J_3(\mathbb{C})$. We are particularly interested in idempotents for the exceptional algebra $J_3(\mathbb{O})$ over the octonions [30][31], and its extensions by an arbitrary division algebra. Its off diagonal integral points are known to give the Leech lattice [32][33].

The four dimensional Fourier transform is the eigenvectors of the chiral operator γ_5 in the Dirac representation,

$$F_4 = F_2 \otimes F_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{24}$$

As Alain Connes likes to say, our ignorance of numbers comes down to the interplay between multiplication and addition on the adelic line. Discreteness in the golden ring integers tells us to reconsider all valuations, which we do by introducing coordinates in terms of geometrical elements, as required in any motivic approach.

3. Operads, logic and the magic star

Scattering amplitudes using on-shell methods utilise combinatorial operads, most notably Stasheff's associahedra [34][35][22]. Word labels are given to finite rooted planar trees in the following way. Given l letters, the set of words of length n define the vertices of a subdivided cube. For example, when $n = l = 3$ we have a cube in dimension 3 such that each edge is cut in half, giving the vertices of a cuboctahedron. Words are noncommutative monomials. The commutative monomials define triangular simplices, sliced diagonally through the corresponding cube.

Postnikov [36] defines the associahedra and permutohedra in terms of these integral simplex coordinates, which we view as powers of prime components in the factorization of an integer. For the vertices of the permutohedron: write down all divisors of the number $N = {p_1}^n {p_2}^{n-1} {p_3}^{n-2} \cdots p_n$ including 1. For the 24 permutations of S_4, we get

$$\begin{aligned} 1, p_1, p_2, p_3, {p_1}^2, {p_2}^2, {p_1}^3, \\ p_2p_3, p_1p_3, p_1p_2, p_3{p_2}^2, p_3{p_1}^2, p_2{p_1}^2, p_1{p_2}^2, p_2{p_3}^2, \\ p_2{p_3}^3, {p_2}^2{p_3}^3, {p_2}^2{p_3}^2, p_1p_2p_3, \\ p_1{p_2}^2p_3, p_1{p_2}^2{p_3}^2, p_1{p_2}^2{p_3}^3, p_1p_2{p_3}^2, p_1p_2{p_3}^3. \end{aligned} \tag{25}$$

The vertices of the associahedron are obtained by looking at divisor classes: two divisors are equivalent if their set of powers is the same. In the case of S_4 we obtain the 14 vertex associahedron

$$\begin{aligned} 1, p_1, {p_1}^2, {p_1}^3, p_1p_2, {p_1}^2p_2, {p_1}^3p_2, {p_1}^3{p_2}^2, {p_1}^2{p_2}^2, \\ p_1p_2p_3, {p_1}^2p_2p_3, {p_1}^2{p_2}^2p_3, {p_1}^3p_2p_3, {p_1}^3{p_2}^2p_3, \end{aligned} \tag{26}$$

alternatively written as

$$000, 100, 200, 300, 110, 210, 310, 320, 220, 111, 211, 221, 311, 321. \tag{27}$$

Finally, the vertices of Kapranov's permutoassociahedron [37] are pairs $(u)[v]$, with u in S_n and v in the class for the associahedron. For example, write $(p_1{}^2)[q_1q_2]$ to denote $p_1{}^2 \in S_3$ and the class of p_1p_2. The 120 vertex polytope contains 24 pentagons, much like the discrete Hopf fibration [38] for $S^7 \to S^4$, related to the $\mathbf{e_8}$ lattice. Extending rooted trees in both the upwards and downwards directions (in a PRO) we obtain a $(3+3)$ dimensional catalog of the 240 roots of $\mathbf{e_8}$.

Consider now the magic star for $\mathbf{e_8}$. A basic parity cube for three qubits ($n = 3$, $l = 2$) is inscribed inside larger cubes, starting with the cube with halved edges, for the $\mathbf{d_3}$ lattice. The magic star projection [39][40][41] of Lie algebra lattices contains a hexagon of edge centers on the cube, perpendicular to the three points on the diagonal, say $(0,0,0)$, $(1,1,1)$ and $(2,2,2)$. This diagonal collapses to the central $\mathbf{e_6}$ point in the magic plane.

This magic plane is tiled by a tetractys simplex for 3 qutrits, since the triangles defining the star of David contain two interior points along each edge. The associahedra sit inside simplices of this type, where $n = l$. For the tetractys, we obtain Postnikov's pentagon [36] as the five points (x_1, x_2, x_3) with $\sum_i x_i = 3$ such that in the i-th coordinate the sum of x_i with $i \leq i$ is greater than or equal to i. These coordinates directly encode the noncommutative forest representation of vertices on the associahedron. Now one tetractys holds exactly three pentagons.

A generic point on the star hexagon in the magic plane [39] carries two labels:

(i) one of the 27 dimensions of $J_3(\mathbb{O})$ for $\mathbf{e_8}$ (or its analogue for other algebras)

(ii) a word label for the center of a tetractys tile, out of the six elements of S_3.

Now as it happens, $6/27 = 2/9$ is very close to the θ parameter for the lepton Koide mass matrices [17][18][16][42], something that certainly warrants further investigation. Moreover, in Tesla numerology it is observed that ϕ^2 is close to $5\pi/6$, which appears in a knot invariant estimate for the fine structure constant [42].

A discrete blowup, or scale shift, sends each point in the plane to a new tetractys, creating larger copies of the magic star. Three dimensional space is regularly tiled either by parity cubes or by permutohedra. The parity cube is derived from the permutohedron as follows. The 24 vertices of S_4 have integral coordinates, namely the permutations of $(1,2,3,4)$ in $\mathbb{R}^3$. Each $s \in S_4$ is assigned a *signature*, which lists the shifts between entries in s. For example, $(2,3,1,4)$ has signature $+-+$. The eight signature classes for S_4 define the vertices of the parity cube in three dimensions. The product of two signature classes for S_n is given by the product in the group Hopf algebra $\mathbb{C}S_n$, producing the descent Hopf algebra of Solomon [43]. For example,

$$(-++)(--+) = (+--) + (+-+) + (---). \tag{28}$$

In any dimension, this is the Hopf algebra needed to construct Jordan pairs [44][45][46] for the magic star.

In the Jordan pair picture, points and higher dimensional objects are idempotents. Similarly, from a categorical perspective, open sets are idempotents in a Heyting algebra [47], which is a not necessarily distributive poset lattice, with 0, 1 and an implication map $x \Rightarrow y$. Objects in the lattice satisfy

$$x \vee x = x, \quad x \wedge x = x \tag{29}$$

and

$$x \wedge (y \vee x) = x = (x \wedge y) \vee x. \tag{30}$$

Implication satisfies

$$(x \Rightarrow y) \wedge x = x \wedge y \tag{31}$$

and the higher distributivity rule

$$x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z). \tag{32}$$

A lattice is non distributive if it contains a pentagon. Vector spaces for quantum logic are non distributive under the closure of union operation.

The loop structure on octonions in the $\mathbf{e_8}$ lattice weakens both commutativity and associativity, effectively introducing braiding and fusion rules for a category.

The duality of product and coproduct appears on the fundamental cubes. The α_i labels of (17) give the parity cube as a three element power set lattice. Conversely, when we consider instead a quantum configuration of three non parallel lines in a plane, the source (the empty set ν) becomes the whole plane, while the target three point set is now the empty intersection of the three lines.

4. Masses and Mixings

If there are three rest mass particle states, and mass is energy, then the limit of the uncertainty principle requires three time lines. In the neutrino CMB correspondence, temperatures correspond to past, present and future [6]. The breaking of the massless 3×3 matrix spectrum requires a braiding, associated to dyonic states, and a quantum neutrino mass.

A rest mass operator $\sqrt{M}$ obeying the Koide relation [16][17][18] is a circulant of the form (22) plus a scale factor μ. The diagonal of eigenvalues under F_3 gives a sum of three idempotents for $J_3(\mathbb{C})$, where we recall that $\mathbb{C}$ is densely filled by $\mathbb{Z}^8/2$ using the quasilattice of [1]. The determinant of $\sqrt{M}$ [42] satisfies

$$1 + \sqrt{\lambda_1 \lambda_2 \lambda_3} \equiv r \cos(3\theta), \tag{33}$$

and this quantity goes to zero when one eigenvalue of $\sqrt{M}$ is negative, which occurs for Brannen's extension [16] of the Koide rule to the neutrinos. This justifies the choice $r = \sqrt{2}$ for the leptons, since it is centered around zero for the basic arithmetic phase $\theta = \pm\pi/12$. This rule provides a second constraint on the mass triplet, after the Koide relation itself. For neutrinos, it takes the form

$$\frac{\sqrt{m_1 m_2 m_3}}{(\sqrt{m_1} + \sqrt{m_2} - \sqrt{m_3})^3} = \frac{1}{27}. \tag{34}$$

Observationally, the charged lepton phase θ is very close to $2/9$, while the active neutrino triplet fits oscillation data with a phase of $\theta = 2/9 + \pi/12$ [9]. Similarly, quark rest mass triplets are obtained with phases $2/27$ and $4/27$. The charged lepton scale μ is a simple multiple of the proton mass [16] and the right handed neutrino phase is $2/9 - \pi/12$ [6].

Although a better clarification of mixing angles relies on further details in motivic quantum gravity, we outline here a useful quark lepton complementarity model. The first Euler angle in the CKM matrix [48][49] is the Cabibbo angle δ_{12}, approximated by the rule

$$\delta_{12} + \delta_{13} = \frac{\pi}{4} - \arctan\frac{1}{\phi} = 13.28^\circ. \tag{35}$$

The other two irrationals in the golden ring give

$$\delta_{23} = \frac{\pi}{6} - \arctan\frac{1}{\sqrt{\phi+2}} = 2.3^\circ, \;\; \frac{\pi}{2} - \arctan\frac{1}{\phi\sqrt{\phi+2}} = 72^\circ, \tag{36}$$

where 72° is an angle in the Penrose tiling [1][50][51]. The phases $\pi/6$ and $\pi/4$ define the tribimaximal matrix [52], which is a planar approximation to the PMNS neutrino mixing matrix [53][54]. For the small quark phase $\delta_{13} \simeq 0.2^\circ$ we look to the breaking of tribimaximal mixing in the neutrino sector. The small mixing angle 8.5° is close to $4/27$, which is obtained as two thirds of $2/9$ from a triality action on the complex phase X in

$$\begin{pmatrix} a & X & \overline{X} \\ \overline{X} & a & X \\ X & \overline{X} & a \end{pmatrix} \tag{37}$$

in $J_3(\mathbb{C})$ [42]. With this deformation, the CKM matrix $\delta_{12} = 13.01°$ and δ_{13} is close to 0.27. Since Euler angles are expressed as circulants in the Hopf algebra $\mathbb{C}S_3$, complex phases are automatically included and the CKM and PMNS matrices exhibit maximal CP violation, in line with observations.

In the full theory, the phases $\pi/6$ and $\pi/4$ are of course associated to automorphic forms, which we construct from the combinatorics of higher dimensional discrete geometries. Here, for instance, the allowed dimensions of matrix components in the 3×3 T-algebras of [41] determine the restricted Euler factors, such as

$$\begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & a+b \end{pmatrix}, \tag{38}$$

of our circulant mixing matrices.

Acknowledgments

The author thanks Michael Rios, Piero Truini, Alessio Marrani and Ray Aschheim for numerous discussions and also for their participation in Group32. Thanks also to the team at Quantum Gravity Research in Los Angeles.

References

[1] Battaglia F and Prato E 2010 *Commun. Math. Phys.* **299** 577
[2] Coecke B and Abramsky S 2004 *Proc. 19th Conf. on Logic in Computer Science (Turku)* (IEEE) p 415
[3] Coecke B, Heunen C and Kissinger A 2016 *Quant. Info. Process.* **15** 5179
[4] Dungworth G 2010 Galaxy Zoo forums *Preprint* Classified or Deleted
[5] Sheppeard M D 2010 Arcadian functor *Preprint* kea-monad.blogspot.com
[6] Sheppeard M D 2017 The algebra of non local neutrino gravity *Preprint* vixra:1712.0076
[7] Dvali G and Funcke L 2016 *Phys. Rev.* D **93** 113002
[8] Dungworth G and Sheppeard M D 2017 Non local mirror neutrinos with $R = ct$ *Preprint* vixra:1711.0119
[9] de Salas P F, Forero D V, Ternes C A, Tortola M and Valle J W F 2018 Status of neutrino oscillations *Preprint* arXiv:1708.01186
[10] McCulloch M E 2012 *Astrophys. and Space Science* **342** 575
[11] Gine J 2012 *Mod. Phys.* A **27** 1250208
[12] McCulloch M E and Gine J 2017 *Mod. Phys.* A **32** 1750148
[13] Gine J 2011 *Int. J. Theor. Phys.* **50** 607
[14] Bilson-Thompson S O 2005 A topological model of composite preons *Preprint* hep-ph/0503213
[15] Sheppeard M D 2010 Quark lepton braids and heterotic supersymmetry *Preprint* vixra:1004.0083
[16] Brannen C A 2006 Lepton masses *Preprint* www.brannenworks.com
[17] Koide Y 1983 *Phys. Rev.* D **28** 252
[18] Koide Y 1983 *Phys. Lett.* B **120** 161
[19] Furey C 2014 *JHEP* **10** 046
[20] Furey C 2012 *Phys. Rev.* D **86** 025024
[21] Furey C 2015 *Phys. Lett.* B **742** 195
[22] Sheppeard M D 2007 Gluon phenomenology and a linear topos *PhD thesis* University of Canterbury
[23] Tesla N 2015 *The True Wireless* (USA: Simon and Schuster)
[24] Freedman M, Larsen M and Wang Z 2002 *Commun. Math. Phys.* **227** 605
[25] Levin M A and Wen X G 2005 *Phys. Rev.* B **71** 045110
[26] Gresnigt N G 2018 *Phys. Lett.* B **783** 212
[27] Schwinger J 1960 *Proc. Nat. Acad. Sci. USA* **46** 570
[28] Wootters W K and Fields B D 1989 *Ann. Phys.* **191** 363
[29] Combescure M 2009 *J. Math. Phys.* **50** 032104
[30] Baez J 2002 *Bull. Amer. Math. Soc.* **39** 145
[31] McCrimmon K 2003 *A Taste of Jordan Algebras* (Berlin: Springer)
[32] Wilson R A 2009 *J. Alg.* **322** 2186
[33] Rios M 2013 *Preprint* arXiv:1307.1554
[34] Stasheff J D 1963 *Trans. Amer. Math. Soc.* **108** 293

[35] Brown F C S 2009 *Annal. Scient. de l'Ecole Normale Superieure* **42** 371
[36] Postnikov A 2009 *Int. Math. Res. Not.* **2009** 1026
[37] Kapranov M 1993 *J. Pure App. Alg.* **85** 119
[38] Sadoc J F and Mosseri R 1993 *J. Non-crystalline Solids* **153** 247
[39] Truini P 2012 *Pacific J. Math.* **260** 227
[40] Marrani A and Truini P 2016 *P-adic Numbers, Ultrametric Anal. and App.* **8** 68
[41] Truini P, Rios M and Marrani A 2017 The magic star of exceptional periodicity *Preprint* arXiv:1711.07881
[42] Sheppeard M D 2017 Lepton mass phases and the CKM matrix *Preprint* vixra:1711.0336
[43] Loday J L and Ronco M O 1998 *Adv. Math.* **139** 293
[44] Truini P and Biedenharn L C 1982 *J. Math. Phys.* **23** 1327
[45] Faulkner J R 2000 *J. Alg.* **232** 152
[46] Faulkner J R 2004 *J. Alg.* **279** 91
[47] MacLane S and Moerdijk I 1994 *Sheaves and Geometry in Logic* (New York: Springer)
[48] Cabibbo N 1963 *Phys. Rev. Lett.* **10** 531
[49] Kobayashi M and Maskawa T 1973 *Prog. Theor. Phys.* **49** 652
[50] Irwin K, Amaral M M, Aschheim R and Fang F 2016 *Proc. 4th Int. Conf. on the Nature and Ontology of Spacetime (Varna)* (Minkowski Institute) p 117
[51] Aschheim A and Rios M 2018 *Preprint* in this volume
[52] Harrison P F, Perkins D H and Scott W G 2002 *Phys. Lett.* B **530** 167
[53] Pontecorvo B 1958 *Sov. Phys. JETP* **7** 172
[54] Maki Z, Nakagawa M and Sakata S 1962 *Prog. Theor. Phys.* **28** 870