

The Domain of the Riemann Zeta Function on the Complex Plane

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ABSTRACT

The Riemann zeta-function is one of the most studied complex function in mathematics. Riemann in his 1859 paper *On the Number of Prime Numbers less than a Given Quantity* [1] claimed that the analytic continuation of the zeta-function extends its domain over the entire complex plane except at $s = 1$. But, as I've shown in my two papers: *A Short Disproof of the Riemann Hypothesis* [2] and *Riemann's Functional Equation is Not a Valid Function and Its Implication on the Riemann Hypothesis* [3], the zeta-function is only defined on the right half-plane. It is the purpose of this present work to precisely locate the domain of this function on the right half-plane.

The Riemann Zeta-Function

The Riemann zeta-function is given by the infinite series

$$(1) \quad \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s (= \sigma + \omega i)$ is the complex variable with real part σ and imaginary part ω , that is,

$$\Re(s) = \sigma \quad \text{and} \quad \Im(s) = \omega,$$

respectively, and i is the imaginary unit equal to $\sqrt{-1}$. A positive real number associated with any complex quantity is known as its modulus, usually denoted by $|\zeta(s)|$. The quantity σ has a dampening effect on $\zeta(s)$ while ω acts as a filter that can remove some of its components. Thus, the values of σ and ω can have a great effect on the convergence of the infinite series in (1).

If $\sigma > 1$ and for every ω , the series converges absolutely

$$|\zeta(\sigma + \omega i)| = \left| \sum_{n=1}^{\infty} n^{-\sigma + \omega i} \right| \leq 1 + 2^{-\sigma} + 3^{-\sigma} + 4^{-\sigma} + 5^{-\sigma} + \dots + n^{-\sigma} = \sum_{n=1}^{\infty} n^{-\sigma},$$

while if $0 < \sigma \leq 1$ and $\omega \neq 0$, the series is said to be conditionally convergent. It may converge if σ is large enough. But, how large?

If the series in (1) is expressed as

$$\zeta(s) = \sum_{n=1}^{\infty} e^{-s \log n} = \sum_{n=1}^{\infty} e^{-\sigma \log n - i \omega \log n} = \sum_{n=1}^{\infty} n^{-\sigma} (\cos \omega \log n - i \sin \omega \log n),$$

where $\log n$ is the natural logarithm of n ; then, its modulus $|\zeta(\sigma + \omega i)|$ is

$$|\zeta(\sigma + \omega i)| = \sqrt{\left(\sum_{n=1}^{\infty} n^{-\sigma} \cos \omega \log n \right)^2 + \left(\sum_{n=1}^{\infty} n^{-\sigma} \sin \omega \log n \right)^2},$$

or,

$$(2) \quad |\zeta(\sigma + \omega i)| = \sqrt{\sum_{n=1}^{\infty} n^{-2\sigma} + 2 \sum_{n=2}^{\infty} n^{-\sigma} \cos \omega \log n + 2 \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} n^{-\sigma} (n+k)^{-\sigma} \cos \omega \log \left(\frac{n+k}{n} \right)}.$$

One can also express (2) as

$$|\zeta(\sigma + \omega i)| = \sqrt{S_1 + S_2 + S_3} = \sqrt{S},$$

where,

$$S_1 = \sum_{n=1}^{\infty} n^{-2\sigma}, \quad S_2 = 2 \sum_{n=2}^{\infty} n^{-\sigma} \cos \omega \log n, \quad S_3 = 2 \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} n^{-\sigma} (n+k)^{-\sigma} \cos \omega \log \left(\frac{n+k}{n} \right),$$

and $S = S_1 + S_2 + S_3$.

The first series S_1 is independent of ω and converges absolutely if $\sigma > 1/2$; the second series S_2 converges absolutely if $\sigma > 1$; and the double sum S_3 converges quickly due to its highly damped coefficients. The value of the sum S must be greater than or equal to zero (i.e., $S \geq 0$) in order for (1) to be valid; since if $S < 0$, the modulus $|\zeta(\sigma + \omega i)|$ will not be a real number.

It is known that $|\zeta(\sigma + \omega i)|$ is always greater than zero if $\sigma > 1$ and for every ω . The sum $S_1 + S_2$, will converge to a value say, A , and is unlikely to reduce $|\zeta(\sigma + \omega i)|$ to zero due to S_1 providing a fixed positive value for a given value of σ : that is, $S_1 > |A|$ such that

$$|\zeta(\sigma + \omega i)| = \sqrt{S_1 + A} > 0. \quad \text{Hence, the zeta function has no zeros if } \sigma > 1.$$

If $1/2 < \sigma \leq 1$, S_1 converges absolutely and the sum $S_1 + S_2$ may converge for some $\omega \neq 0$. Hence, $|\zeta(\sigma + \omega i)|$ is conditionally convergent if $1/2 < \sigma \leq 1$ and $\omega \neq 0$. One can also conclude that the zeta function has no zeros in this region for the same reason given above.

If $0 < \sigma \leq 1/2$ and for every ω , S_1 diverges and it doesn't matter whether the sum $S_1 + S_2$ converge or not, the zeta function is not defined on this region. It is also interesting to note that $\zeta\left(\frac{1}{2} + \omega i\right)$ is **undefined** on this region which proves that the Riemann Hypothesis is false.

If $\sigma \leq 0$ and for every ω , S_1 diverges very fast and $\zeta(s)$ is not defined on this region.

Therefore, the **domain of $\zeta(s)$ is on $\sigma > 1/2$**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left\{ \begin{array}{l} \frac{1}{2} < \sigma \leq 1 \text{ and } \omega \neq 0, \\ \sigma > 1. \end{array} \right.$$

It has no zeros in its entire domain and its modulus is always greater than zero,

$$|\zeta(s)| > 0 \left\{ \begin{array}{l} \text{If } \frac{1}{2} < \sigma \leq 1 \text{ and } \omega \neq 0, \\ \text{If } \sigma > 1 \text{ and for every } \omega. \end{array} \right.$$

REFERENCES

- [1] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity*.
- [2] Evangelista, Armando M. (2018). *A Short Disproof of the Riemann Hypothesis*.
<http://vixra.org/pdf/1808.0599v1.pdf>
- [3] Evangelista, Armando M. (2018). *Riemann's Functional Equation is Not a Valid Function and Its Implication on the Riemann Hypothesis*. <http://vixra.org/pdf/1808.0624v1.pdf>

LINKS:

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis