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Bifurcations and the Dynamic Content of Particle Physics

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Abstract

We have recently conjectured that the flow from the ultraviolet (UV) to the infrared (IR) sector of any multivariable field theory approaches chaotic dynamics in a universal way. A key assumption of this conjecture is that the flow evolves in far-from-equilibrium conditions and it implies that the end-point attractor of effective field theories replicates the geometry of multifractal sets. Our conclusions are further reinforced here in the framework of *nonlinear dynamical systems* and *bifurcation theory*. In particular, it is found that steady-state perturbations near the IR attractor induce formation of Dark Matter structures while oscillatory perturbations lead to the field content of the Standard Model.

Key words: Bifurcations, Dynamical Systems, Strange Attractors, Center Manifold Theory, Normal Forms, Standard Model, Cantor Dust.

1. Introduction and motivation

The Renormalization Group (RG) is a well-established framework for the analysis of complex physical systems at both ends of the energy scale. Over the years, the principles and methods of RG have found a wide range of applications, from critical behavior in Statistical Physics and Condensed Matter to perturbative and non-perturbative models in Quantum Field Theory (QFT) [7, 26, 42]. An appealing feature of RG equations is that they resemble the *evolution equations of dynamical systems* [16-17, 20]. In particular, the Callan-Symanzik equation stems from the independence of QFT from its subtraction point, which is on par with self-similarity of autonomous flows approaching attractors. In the Wilsonian formulation of the RG, the flow in coupling space is associated with the trajectory of QFT towards a subspace of *relevant* and *marginal* operators. Conventional wisdom asserts that the attractors of the RG flow consist of a finite number of isolated fixed points (FP). There is now mounting evidence that this assumption is too restrictive, that RG flows – echoing the onset of turbulence in fluid mechanics – may evolve towards limit cycles or tori as well as *strange attractors*, the latter denoting invariant sets having chaotic structure [20, 38-41].

The goal of this work is to extrapolate the conventional RG paradigm to a framework which minimizes the potential loss of generality due to simplifying assumptions. To this end, we posit that all trajectories connecting the UV and IR sectors of a *generic field theory* are characterized by the following initial conditions:

a) a large count of independent or coupled variables,
b) a large count of independent or coupled control parameters,
c) far-from-equilibrium settings,
d) non-perturbative and non-integrable dynamics.

In our view, the motivation for this extended framework is that the combined use of a) to d) enable a more realistic picture of *complex dynamics* that is likely to define the UV to IR flow. This view is backed up by many examples. For instance, integrable dynamical systems are isomorphic to free, non-interacting theories, which are unable to account for the arrow of time in transient regimes, the physics of self-organization and complex evolution outside equilibrium [19, 24-25]. Another instance is provided by Sakharov's non-equilibrium conditions for baryogenesis and the observed baryon asymmetry of the Universe [35].

Our paper is organized as follows: the interpretation of RG flows as autonomous dynamical systems is detailed in section 2. Section 3 delves into the universal theory of flows evolving in far-from-equilibrium conditions and their reduction to *normal form equations*. The bifurcations generated by these equations and their connection to the structure of SM and Dark Matter form the topic of next three sections. Conclusions are summarized in the last section.

We caution the reader on the introductory and tentative nature of this work. Our sole goal is to draw attention to the many unexplored implications of nonlinear science and complexity theory on the dynamics of the SM and beyond. Additional research is needed to reject or expand the body of ideas discussed here.

2. RG flows as autonomous dynamical systems

The RG flow in the space of couplings $g \in \Gamma$ is a continuous map $\beta = R \times \Gamma \to \Gamma$ called the "beta function" and associated with [20]

$$\beta(g) = \mu \frac{dg}{d\mu} = \frac{dg}{d(\log \mu/\mu_0)} \tag{1}$$

such that

$$\beta(0, g) = g \tag{2}$$

$$\beta(\tau, \beta(s, g)) = \beta(\tau + s, g) \tag{3}$$

where the "RG time" is $\tau = \log(\mu / \mu_0)$ and μ is the RG scale. A FP (equilibrium or conformal point) of (1) is a coupling $g_0 \in \Gamma$ for which $\beta(R, g_0) = g_0$. The FP of the RG flow correspond to zero or infinite correlation lengths and are accordingly classified as "*trivial*" or "*non-trivial*". The existence of FP reflects the asymptotic approach towards *scale-invariance* and it relates to the self-similarity of fractal structures [see e.g. 7]. A subset $\mathrm{I} \subset \Gamma$ is an invariant set of the flow if

$$\beta(R, \mathrm{I}) = \bigcup_{t \in \mathbf{R}} \beta(\tau, \mathrm{I}) \subset \Gamma \tag{4}$$

Likewise, the continuous time flow of autonomous dynamical systems is described by the differential equation

$$\frac{dx(\tau)}{d\tau} = f(x(\tau)) \tag{5}$$

where $x \in R^n$ and $f : R^n \to R^n$ is a function on the n-dimensional phase space R^n [4]. There are two ways of relating (5) to a *map iteration* of the phase space onto itself, namely,

a) Working in discrete time ($\tau \to \tau_0$) turns (5) into

$$x_{n+1} = x_n + \tau_0 f(x_n) = F(x_n) \ , \ \ x_n = x(n\tau_0) \tag{6}$$

b) If (1) has periodic solutions $x(T) = x(0) = x_0$ for some $T > 0$, one takes a hyperplane R^{n-1} of dimension $n-1$ transverse to the orbit $\tau \to x(\tau)$ through x_0 and evaluates the distribution of neighboring intersections of the orbit with this hyperplane (the method of Poincaré sections).

Many dynamical systems and maps are dependent on a number of control parameters $\lambda \in R^m$. In this case, (5) and (6) take the form

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), \lambda(\tau)) \tag{7}$$

$$x_{n+1} = x_n + \tau_0 f(x_n, \lambda) = F(x_n, \lambda) \tag{8a}$$

Of particular interest is the long-term evolution of (6)-(8), which reflects the behavior of the large k th iterate of the flow in phase-space, $\{F^k(x)\}$, $k >> 1$. By definition, a *period-k* FP of map (6) satisfies the condition

$$x^*_{n+k} = F^{(k)}(x^*_n) = x^*_n \tag{8b}$$

Some flows may converge to specific attractors like a FP or a periodic orbit or erratically wander inside a bounded region (I). If all iterates remain "trapped" in (I) for $x \in \mathrm{I}$, then (I) forms an *invariant set* [5-6]. Moreover, if (I) has a fine structure, or if there is sensitive dependence on initial conditions (two nearby points get farther apart under a large number of iterates of f), then (I) represents a *strange set*.

3. Flows in far-from-equilibrium field theory

Quantum Field Theories are known to become scale-invariant at large distances. Viewed in the context of *conformal field theory*, this property is typically associated with the FP structure of the RG flow [7, 26, 42]. Starting from this observation, we conjecture below that all field theories evaluated at sufficiently low-energy scales emerge from an underlying system of high-energy entities called *primary variables*. Let the UV sector of field theory be described by a large set of such variables $x \equiv \{x_i\}$, $i = 1, 2, ..., n$, $n >> 1$, whose mutual coupling and dynamics is far-from-equilibrium. The specific nature of the UV variables is irrelevant to our context, as they can take the form of irreducible objects such as, but not limited to, *spinors, quaternions, twistors, strings, branes, loops, knots, bits of information* and so on.

The downward flow of $x \equiv \{x_i\}$ may be mapped to a system of ordinary differential equations having the universal form

$$x'_\tau = f(x(\tau), \lambda(\tau), D(\tau)) \tag{9}$$

Here, λ, τ, D denote, respectively, the control parameters vector $\lambda = \{\lambda_j\}$, $j = 1, 2, ...m$, the evolution parameter and the dimension of the embedding space. If the dimension of the embedding space is taken to be independent variable or control parameter, the system (9) further reduces to

$$x'_\tau = f(x(\tau), \lambda(\tau)) \tag{10}$$

It is reasonable to assume that the flow (9) or (10) occurs in the presence of non-vanishing *perturbations* induced by far-from-equilibrium conditions. These may surface, for example, from primordial density fluctuations in the early Universe or from unbalanced vacuum fluctuations in the UV regime of QFT.

To make explicit the effect of perturbations, we resolve $x(\tau)$ into a reference stable state $x_s(\tau)$ and a deviation generated by perturbations, i.e.,

$$x(\tau) = x_s(\tau) + y(\tau) \tag{11}$$

Direct substitution in (10) yields the set of homogeneous equations

$$y'_{\tau} = f(\{x_s + y\}, \lambda) - f(\{x_s\}, \lambda) \tag{12}$$

Further expanding around the reference state leads to

$$y'_{\tau} = \sum_j L_{ij}(x_s, \lambda) y_j + h_i(\{y_j\}, \lambda) \tag{13}$$

where L_{ij} and h_i denote, respectively, the coefficients of the linear and nonlinear contributions induced by departures from the reference state. Here, L_{ij} represents a $n \times n$ matrix dependent on the reference state and on the control parameters vector. Under the assumption that parameters λ stay close to their critical values $(\lambda = \lambda_c)$, it can be shown that either (1) or (2) undergoes bifurcations and they can be mapped to a closed set of universal equations referred to as *normal forms* [1-3]. If, at $\lambda = \lambda_c$ perturbations are non-oscillatory (steady-state), the normal form equations are

$$z'_{\tau} = (\lambda - \lambda_c) - uz^2 \tag{14a}$$

$$z'_{\tau} = (\lambda - \lambda_c)z - uz^3 \tag{14b}$$

$$z'_{\tau} = (\lambda - \lambda_c)z - uz^2 \tag{14c}$$

Instead, if perturbations are oscillatory at $\lambda = \lambda_c$, the normal form equation is given by

$$z'_{\tau} = [(\lambda - \lambda_c) + i\omega_0]z - uz|z|^2 \tag{15}$$

where ω_0 is the frequency of perturbations at the bifurcation point and both u and z are complex-valued. It can be shown that (15) belongs to a rich spectrum of Andronov-Hopf bifurcation scenarios involving *limit cycles* [1-3, 27-33].

We end the section with the following observation: of particular interest is to augment the conditions a) to c) of section 1 with the assumption that (9) and (10) exhibit *memory* effects. These effects may be naturally attributed to a non-local dynamic regime that is far-from-equilibrium and whose characterization requires *fractional calculus* instead of ordinary calculus on smooth manifolds [43, 44]. It is reasonable to conjecture that (9) and (10) evolve in low-fractality conditions defined by arbitrarily small deviations from four-dimensionality of ordinary spacetime [ ]. Under these assumptions, the condition $\varepsilon = 4 - D << 1$ describes the *minimal fractal manifold* (MFM) configuration of spacetime near the IR attractor of (9) and (10), whereby ε takes on the role of leading control parameter [9]. One then naturally proceeds with the identification $\varepsilon \Rightarrow \lambda$, $\varepsilon_c = 0 \Rightarrow \lambda_c$ in (14) and (15), which shows that the four-dimensional spacetime represents the asymptotic limit of the MFM at the critical point $D = 4$.

In summary, the outcome of this analysis is that the multivariable dynamics (9) and (10) reduces in the long-run to a lower dimensional system of *universal equations* with the emerging variable z playing the role of an effective order parameter. If, in addition, (9) and (10) carry low-amplitude non-local effects, the leading control parameter near the IR attractor may be assumed to be $\varepsilon = 4 - D << 1$.

<u>4. Universal bifurcations of the normal form equations</u>

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5. Flavor replication in the Standard Model

The paradigm outlined above hints that the family structure of the Standard Model unfolds from letting (15) develop *sequential bifurcations*. One possible scenario is that the gluon octet emerges as twofold replica of the electroweak boson quartet as in

$$\begin{pmatrix}\gamma & W^{+} & W^{-} & Z\end{pmatrix} \rightarrow (g_i)_{i=1,2,...,8} \tag{1.1}$$

Likewise, color quartets may surface as twofold replicas of lepton doublets, namely,

$$\begin{pmatrix}\nu_e & e\end{pmatrix} \rightarrow \begin{pmatrix} u_R & d_R \\ u_G & d_G \end{pmatrix} \tag{1.2}$$

$$\begin{pmatrix}\nu_\mu & \mu\end{pmatrix} \rightarrow \begin{pmatrix} c_R & c_R \\ s_G & s_G \end{pmatrix} \tag{1.3}$$

$$\begin{pmatrix}\nu_\tau & \tau\end{pmatrix} \rightarrow \begin{pmatrix} t_R & t_G \\ b_R & b_G \end{pmatrix} \tag{1.4}$$

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Colorless constraint

$$R + G + B = 1 \tag{1.5}$$

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Finally, the Higgs scalar arises as topological condensate of gauge bosons having anti-parallel spins. The simplest combination of weakly-coupled gauge bosons condensing into a spin-zero state is given by [ ]

$$\Phi_c = \frac{1}{4}[(W^+ + W^- + Z + \gamma + g) + (W^+ + W^- + Z + \gamma + g)] \tag{1.6}$$

The number of SM flavors is constrained by anomaly cancelation [ ] and by the closure relationship [ ],

$$\sum_{j=1}^{16} (\frac{m_i}{M_{EW}})^2 = 1 \tag{1.7}$$

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One can argue that it may also be fixed by demanding marginal stability of the perturbative RG flow [ ].

6. Cantor Dust as underlying content of Dark Matter

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7. Concluding remarks

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