The Riemann Transform

By

Armando M. Evangelista Jr.

armando781973@icloud.com

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ABSTRACT

In his 1859 paper, Bernhard Riemann used the integral equation \( \int_0^\infty f(x) x^{-s-1} \, dx \) to develop an explicit formula for estimating the number of prime numbers less than a given quantity. It is the purpose of this present work to explore some of the properties of this equation.
Consider the integral equation given below

\[
F(s) = \int_0^\infty f(x) x^{-s-1} \, dx
\]

Formula (1) is the integral of \( f(x) \) times \( x^{-s-1} \) for \( x = 0 \) to \( \infty \) and the resulting function is a function of \( s \), say \( F(s) \) (or the transform of \( f(x) \)). It must be assumed that \( f(x) \) is such that the integral exists (it has finite value).

**Example 1**  Apply formula (1) to obtain the transform of \( f(x) = e^{-x} \).

**Solution.** Substitute \( e^{-x} \) to (1)

\[
F(s) = \int_0^\infty e^{-x} x^{-s-1} \, dx = \Gamma(-s), \quad \Re(s) < 0, \quad \text{since} \quad \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx, \quad \Re(s) > 0,
\]

where \( \Gamma(s) \) is the gamma function and \( \Re(s) \) is the real part of the complex quantity \( s \).

**Unit Step Function (Heaviside Function)**

The unit step function or Heaviside function \( \mu(x-a) \) is 0 for \( x < a \), has a jump size 1 at \( x = a \) (where it is usually consider as undefined), and is 1 for \( x > a \), in a formula:

\[
\mu(x-a) = \begin{cases} 
0 & \text{if } x < a \\
1 & \text{if } x > a
\end{cases} \quad a \geq 0.
\]

The transform of \( \mu(x-a) \) is

\[
F(s) = \int_0^\infty x^{-s-1} \mu(x-a) \, dx = \int_a^\infty x^{-s-1} \, dx = \frac{-x^{-s}}{s} \bigg|_a^\infty ;
\]

here the integration begins at \( x = a \) (>0) because \( \mu(x-a) \) is 0 for \( x < a \). Hence

\[
F(s) = \frac{a^{-s}}{s} \quad (a > 0 \quad \text{and} \quad \Re(s) > 0).
\]

**Example 2:** The Riemann Zeta Function is given by

\[
\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots = \sum_{n=1}^\infty n^{-s} = \sum_{n=1}^\infty \frac{1}{n^s} \quad \Re(s) > 1,
\]
obtain the transform of $\sum_{n=1}^{\infty} \mu(x-n), \quad n=1,2,3,\ldots$

$$F(s) = \int_{0}^{\infty} \left[ \mu(x-1) + \mu(x-2) + \mu(x-3) + \ldots \right] x^{-s-1} dx = \frac{-x^{-s}}{s} \bigg|_{1}^{\infty} + \frac{-x^{-s}}{s} \bigg|_{2}^{\infty} + \frac{-x^{-s}}{s} \bigg|_{3}^{\infty} + \ldots$$

$$= \frac{1}{s} (1+2^{-s}+3^{-s}+4^{-s}+\ldots) = \frac{1}{s} \sum_{n=1}^{\infty} \frac{\zeta(s)}{n^s} = \frac{\zeta(s)}{s}, \quad \Re(s) > 1.$$

**Example 3:** Obtain the transform of $\pi(x) = \sum_{p} \mu(x-p)$, where $p$ is a prime number, $p = 2, 3, 5, 7, 11, \ldots$

$$F(s) = \int_{0}^{\infty} \left[ \sum_{p} \mu(x-p) x^{-s-1} dx \right] = \int_{0}^{\infty} \left[ \mu(x-2) + \mu(x-3) + \mu(x-5) + \mu(x-7) + \ldots \right] x^{-s-1} dx$$

$$\pi(s) = \frac{1}{s} (2^{-s}+3^{-s}+5^{-s}+7^{-s}+\ldots) = \frac{1}{s} \sum_{p} p^{-s} \quad \Re(s) > 1.$$

**Dirac's Delta Function**

Consider the function

$$f_{\tau}(x-a) = \begin{cases} 1/\tau & \text{if } a \leq x \leq a + \tau \\ 0 & \text{otherwise.} \end{cases}$$

Its integral is

$$I = \int_{0}^{\infty} f_{\tau}(x-a) dx = \int_{a}^{a+\tau} \frac{1}{\tau} dx = 1.$$

We let now $\tau$ becomes smaller and smaller and take the limit as $\tau \to 0$ ($\tau > 0$). This limit is denoted by $\delta(x-a)$, that is,

$$\delta(x-a) = \lim_{\tau \to 0} f_{\tau}(x-a).$$

and obtain
\[\delta(x-a) = \begin{cases} \infty & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}\quad \text{and} \quad \int_0^\infty \delta(x-a) \, dx = 1.\]

\(\delta(x - a)\) is called the **Dirac delta function** or the **unit impulse function**. For a continuous function \(f(x)\) one uses the **sifting** property of \(\delta(x-a)\),

\[\int_0^\infty f(x) \delta(x-a) \, dx = f(a).\]

To obtain the transform of \(\delta(x-a)\), we write

\[f_\tau(x-a) = \frac{1}{\tau} \mu(x-a) - \mu(x-(a+\tau))\]

and take the transform

\[F(s) = \int_0^\infty f_\tau(x-a)x^{-s-1} \, dx = \frac{1}{\tau s} \left[ a^{-s} - (a+\tau)^{-s} \right] = \frac{1}{\tau s} \left[ 1 - \frac{(1 + \tau)^{-s}}{a} \right], \quad a > 0 \text{ and } \Re(s) > 0.\]

Take the limit as \(\tau \to 0\). By l’Hôpital’s rule, the quotient on the right has the limit \(1/a\). Hence, the right side has the limit \(a^{-(s+1)}\). The transform of \(\delta(x-a)\) define by this limit is

\[F(s) = \int_0^\infty \delta(x-a)x^{-s-1} \, dx = a^{-(s+1)} \quad a > 0.\]

**Example 4** Obtain the transform of \(\sum_{n=1}^\infty x \delta(x-n)\) and \(\sum_{n=1}^\infty \delta(x-n)\).

\[\int_0^\infty \left( \sum_{n=1}^\infty x \delta(x-n) \right)x^{-s-1} \, dx = \sum_{n=1}^\infty n^{-s} = \zeta(s), \quad \Re(s) > 1,\]

\[\int_0^\infty \left( \sum_{n=1}^\infty \delta(x-n) \right)x^{-s-1} \, dx = \sum_{n=1}^\infty n^{-s(s+1)} = \zeta(s+1), \quad \Re(s) > 0.\]
The Riemann Transform

Many common functions like \( \sin x, \cos x, \ln x, \text{etc.} \), when applied to formula (1) won’t have finite integrals. But if the lower limit for (1) starts at \( x = 1 \), then there are suitable functions such that the integral in (1) exist.

If \( f(x) \) is a function defined for all \( x \geq 1 \), its Riemann transform is the integral of \( f(x) \) times \( x^{s-1} \) for \( x = 1 \) to \( \infty \). It is a function of \( s \), say \( F(s) \), and is denoted by \( R\{f\} \); thus

\[
F(s) = R\{f\} = \int_{1}^{\infty} f(x) x^{s-1} \, dx.
\]

The given function \( f(x) \) in (2) is called the inverse transform of \( F(s) \) and is denoted by \( R^{-1}\{F\} \); that is,

\[
f(x) = R^{-1}\{F\}.
\]

**Example 5** Let \( f(x) = 1 \), find \( F(s) \).

**Solution.** From (2) we obtain by integration

\[
R\{f\} = R\{1\} = \int_{1}^{\infty} x^{s-1} \, dx = \left. -\frac{1}{s} x^{s} \right|_{1}^{\infty} = \frac{1}{s} \quad (\Re(s) > 0).
\]

**Example 6** Let \( f(x) = x^a \), where \( a \) is a constant. Find \( F(s) \).

**Solution.** From (2),

\[
R\{x^a\} = \int_{1}^{\infty} x^a x^{s-1} \, dx = \left. -\frac{1}{s-a} x^{s-a} \right|_{1}^{\infty} = \frac{1}{s-a} \quad (\Re(s-a) > 0).
\]

**THEOREM 1** Linearity of the Riemann Transform

The Riemann transform is a linear operation; that is, for any functions \( f(x) \) and \( g(x) \) whose transforms exist and any constants \( a \) and \( b \) the transform of \( af(x) + bg(x) \) exists, and

\[
R\{af(x) + bg(x)\} = aF(s) + bG(s).
\]
Example 7 Find the transforms of cosh \((a \ln x)\) and sinh \((a \ln x)\).

**Solution.** Since \(\cosh(a \ln x) = \frac{1}{2}(x^a + x^{-a})\) and \(\sinh(a \ln x) = \frac{1}{2}(x^a - x^{-a})\), we obtain from Example 6 and Theorem 1,

\[
R\{\cosh(a \ln x)\} = \frac{1}{2}(R(x^a) + R(x^{-a})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}
\]

\[
R\{\sinh(a \ln x)\} = \frac{1}{2}(R(x^a) - R(x^{-a})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}.
\]

Example 8 Let \(f(x) = x^{\alpha i}\), where \(i\) is the imaginary operator \(i = \sqrt{-1}\). Find \(F(s)\).

**Solution.** From Example 6

\[
R\{x^{\alpha i}\} = \frac{1}{s - \alpha i} = \frac{1}{s - \alpha i} \cdot \frac{s + \alpha i}{s + \alpha i} = \frac{s}{s^2 + \alpha^2} + i \frac{-\alpha \alpha}{s^2 + \alpha^2}.
\]

Example 9 Cosine and Sine

Derive the formulas

\[
R\{\cos(\alpha \ln x)\} = \frac{s}{s^2 + \alpha^2} \text{ and } R\{\sin(\alpha \ln x)\} = \frac{\alpha}{s^2 + \alpha^2}.
\]

**Solution.** From Example 8 and Theorem 1

\[
x^{\alpha i} = \cos(\alpha \ln x) + i \sin(\alpha \ln x)
\]

\[
R\{x^{\alpha i}\} = R\{\cos(\alpha \ln x)\} + i R\{\sin(\alpha \ln x)\}, \text{ thus }
\]

\[
R\{\cos(\alpha \ln x)\} = \frac{s}{s^2 + \alpha^2} \text{ and } R\{\sin(\alpha \ln x)\} = \frac{\alpha}{s^2 + \alpha^2}.
\]

**THEOREM 2 s-Shiftin**g** Theorem**

If \(f(x)\) has the transform \(F(s)\) (where \(s > k\) for some \(k\)), then \(x^a f(x)\) has the transform \(F(s - a)\) (where \(s - a > k\)). In formulas,

\[
R\{x^a f(x)\} = F(s - a)
\]
or, if we take the inverse on both sides

\[ x^a f(x) = R^{-1} [F(s-a)]. \]

**PROOF** We obtain \( F(s-a) \) by replacing \( s \) with \( s-a \) in the integral in (1), so that

\[ F(s-a) = \int_1^\infty x^{-(s-a)-1} f(x) \, dx = \int_1^\infty x^{s-1} [x^a f(x)] \, dx = R[x^a f(x)]. \]

**Example 10** From Example 9 and the \( s \)-Shifting theorem one can obtain the Riemann transform for

\[ R[x^a \cos(\alpha \ln x)] = \frac{s-a}{(s-a)^2 + \alpha^2} \quad \text{and} \quad R[x^a \sin(\alpha \ln x)] = \frac{\alpha}{(s-a)^2 + \alpha^2}. \]

**Existence and Uniqueness of Riemann Transforms**

A function \( f(x) \) has a Riemann transform if it does not grow too fast, say, if for all \( x \geq 1 \) and some constants \( M \) and \( k \) it satisfies

\[ |f(x)| \leq Mx^k. \quad (3) \]

**THEOREM 3 Existence Theorem for Riemann Transforms**

If \( f(x) \) is defined and piecewise continuous on every finite interval on \( x \geq 1 \) and satisfies (3) for all \( x \geq 1 \) and some constants \( M \) and \( k \), then the Riemann transform \( R\{f\} \) exists for all \( s > k \).

**PROOF** Since \( f(x) \) is piecewise continuous, \( x^{s-1} f(x) \) is integrable over any finite interval on the \( x \)-axis,

\[ |R\{f\}| = \left| \int_1^\infty f(x) x^{s-1} \, dx \right| \leq \int_1^\infty |f(x)| x^{s-1} \, dx \leq \int_1^\infty M x^k x^{s-1} \, dx = \frac{M}{s-k}. \]

**Uniqueness.** If the Riemann transform of a given function exists, it is uniquely determined and if two continuous functions have the same transform, they are completely identical.
The transforms of the first and second derivatives of \( f(x) \) satisfy

\[
R(f') = (s+1)F(s+1) - f(1)
\]

\[
R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f''(1)
\]

Formula (4) holds if \( f(x) \) is continuous for all \( x \geq 1 \) and satisfies (3) and \( f' \) is piecewise continuous on every finite interval for \( x \geq 1 \). Formula (5) holds if \( f \) and \( f' \) are continuous for all \( x \geq 1 \) and satisfy (3) and \( f'' \) is piecewise continuous on every finite interval for \( x \geq 1 \).

The proof of (5) now follows by applying integration by parts twice on it, that is

\[
R(f') = \int_1^\infty f'(x)x^{-s-1}dx = [f(x)x^{-s-1}]_1^\infty + (s+1)\int_1^\infty f(x)x^{-s-2}dx = -f(1) + (s+1)F(s+1).
\]

Repeatedly using integration by parts as in the proof of (5) and using induction, we obtain the following theorem.

**Theorem 5** Riemann Transform of the Derivative \( f^{(n)} \) of Any Order

Let \( f, f', \ldots, f^{(n-1)} \) be continuous for all \( x \geq 1 \) and satisfy (2). Furthermore, let \( f^{(n)} \) be piecewise continuous on every finite interval for \( x \geq 1 \). Then the transform of \( f^{(n)} \) satisfies

\[
R(f^{(n)}) = (s+n)(s+n-1)\cdots(s+1)F(s+n) - (s+n-1)(s+n-2)\cdots f(1) -
(s+n-2)(s+n-3)\cdots f'(1) - \cdots - f^{(n-1)}(1).
\]
Example 11 Let \( f(x) = x^2 \). Then \( f(1) = 1, f'(x) = 2x, f'(1) = 2, f''(x) = 2 \). Obtain \( R\{f\}, R\{f'\}, \) and \( R\{f''\} \).

Solution. \( R\{f\} = F(s) = \frac{1}{s-2}, \quad F(s+1) = \frac{1}{s-1}, \quad F(s+2) = \frac{1}{s} \). Hence, by formulas (4) and (5),

\[
R(f') = (s+1)\frac{1}{s-1} - 1 = \frac{2}{s-1}, \quad \text{and} \quad R(f'') = (s+2)(s+1)\frac{1}{s} - (s+1) - 2 = \frac{2}{s}.
\]

Theorem 6 Riemann Transform of Integrals

Let \( F(s) \) denote the transform of a function \( f(x) \) which is piecewise continuous for \( x \geq 1 \) and satisfies formula (3). Then, for \( s > 0, s > k, \) and \( x > 1 \),

\[
(6) \quad R\left\{ \int_1^x f(\tau) \, d\tau \right\} = \frac{1}{s} F(s-1), \quad \text{thus} \quad \int_1^x f(\tau) \, d\tau = R^{-1}\left\{ \frac{1}{s} F(s-1) \right\}.
\]

Proof Let the integral in (6) be \( g(x) \) then \( g'(x) = f(x) \). Since \( g(1) = 0 \) (the integral from 1 to 1 is zero),

\[
R[f(x)] = R[g'(x)] = (s+1)G(s+1) - g(1) = (s+1)G(s+1) = F(s),
\]

replace \( s \) by \( s-1 \), \( ([s-1]+1)G([s-1]+1) = F(s-1) = sG(s) = F(s-1) \).

Division by \( s \) and interchange of the left and right side gives the first formula in (6), from which the second follows.

Example 12 Let \( f(x) = x \). Obtain \( R\{g(x)\} = R\left\{ \int_1^x \tau \, d\tau \right\} = G(s) \).

Solution. \( F(s) = R\{x\} = \frac{1}{s-1}, \quad F(s-1) = \frac{1}{s-2}, \) then \( G(s) = \frac{1}{s(s-2)} \).

Differentiation and Integration of Transforms

Differentiation of Transforms

Given a function \( f(x) \), the derivative \( F'(s) = dF/ds \) of the transform \( F(s) = R\{f\} \) can be obtained by differentiating \( F(s) \) under the integral sign with respect to \( s \). Thus, if
\[ F(s) = \int_{1}^{\infty} f(x) x^{-s-1} \, dx, \quad \text{then} \quad F'(s) = -\int_{1}^{\infty} \ln x \, f(x) x^{-s-1} \, dx. \]

Consequently, if \( R\{f\} = F(s) \), then

\[
R\{\ln x \, f(x)\} = -F'(s) \quad \text{and} \quad R^{-1}\{-F'(s)\} = -\ln x \, f(x),
\]

where the second formula is obtained by applying on both sides of the first formula. In this way, differentiation of a function in the \( s \)-domain corresponds to the multiplication of the function in the \( x \)-domain by \(-\ln x\).

**Example 13** Obtain the transform of \( \ln x \sin(\alpha \ln x) \) and \( \ln x \cos(\alpha \ln x) \).

**Solution.**

\[
R\{\ln x \sin(\alpha \ln x)\} = -\frac{d}{ds}\left[\frac{\alpha}{s^2 + \alpha^2}\right] = \frac{2\alpha s}{(s^2 + \alpha^2)^2},
\]

\[
R\{\ln x \cos(\alpha \ln x)\} = -\frac{d}{ds}\left[\frac{s}{s^2 + \alpha^2}\right] = -\frac{(s^2 + \alpha^2) - 2s^2}{(s^2 + \alpha^2)^2} = \frac{s^2 - \alpha^2}{(s^2 + \alpha^2)^2}. \]

**Integration of Transform**

Given a function \( f(x) \), and the limit of \( f(x)/\ln x \), as \( x \) approaches 1 from the right, exists, then for \( s > k \),

\[
R\left[\frac{f(x)}{\ln x}\right] = \int_{s}^{\infty} F(\sigma) \, d\sigma \quad \text{hence} \quad R^{-1}\left[\int_{s}^{\infty} F(\sigma) \, d\sigma\right] = \frac{f(x)}{\ln x}.
\]

In this way, integration of the transform of a function \( f(x) \) corresponds to the division of \( f(x) \) by \( \ln x \). From the definition it follows that

\[
\int_{s}^{\infty} F(\sigma) \, d\sigma = \int_{s}^{\infty} \int_{1}^{\infty} x^{-\sigma-1} f(x) \, dx \, d\sigma = \int_{s}^{\infty} f(x) \left[\int_{1}^{\infty} x^{-\sigma} \, d\sigma\right] \frac{dx}{x}, \]
Integration of $x^\sigma$ with respect to $\sigma$ gives $x^\sigma/(-\ln x)$. Hence the integral over $\sigma$ on the right equals $x^\gamma/\ln x$. Therefore,

$$\int_s^{\infty} F(\sigma) d\sigma = \int_1^{\infty} x^{-s-1} \frac{f(x)}{\ln x} \, dx = R \left[ \frac{f(x)}{\ln x} \right] \quad (s > k).$$

**Example 14:** Find the inverse transform of $\ln \left( \frac{1 + \alpha^2}{s^2} \right) = \ln \left( \frac{s^2 + \alpha^2}{s^2} \right)$.

**Solution.** Denote the given transform by $F(s)$. Its derivative is

$$F'(s) = \frac{d}{ds} \left[ \ln(s^2 + \alpha^2) - \ln s^2 \right] = \frac{2s}{s^2 + \alpha^2} - \frac{2s}{s^2}.$$

Taking the inverse transform, we obtain

$$R^{-1} F'(s) = R^{-1} \left[ \frac{2s}{s^2 + \alpha^2} - \frac{2s}{s^2} \right] = 2 \cos(\alpha \ln x) - 2 = -\ln x f(x).$$

Hence the inverse $f(x)$ of $F(s)$ is

$$f(x) = \frac{2}{\ln x} [1 - \cos(\alpha \ln x)].$$

Alternatively, if we let

$$G(s) = \frac{2s}{s^2 + \alpha^2} - \frac{2s}{s^2}, \quad \text{then} \quad g(x) = R^{-1} G = -2 [1 - \cos(\alpha \ln x)].$$

From this and using the integral of transform we get,

$$R^{-1} \left[ \ln \frac{s^2 + \alpha^2}{s^2} \right] = R^{-1} \left[ \int_s^{\infty} G(s) ds \right] = -\frac{g(x)}{\ln x} = \frac{2}{\ln x} [1 - \cos(\alpha \ln x)].$$

**The Riemann Transform and the Laplace Transform**

The Laplace transform is the integral of $f(y)$ times $e^{-sy}$ from $y = 0$ to $\infty$ where $f(y)$ is defined for all $y \geq 0$. It is denoted by $L\{f\}$.
(7) \[ L\{f\} = \int_{0}^{\infty} f(y) e^{-sy} dy. \]

The Riemann transform is given below

(8) \[ R\{f\} = \int_{1}^{\infty} f(x) x^{-s-1} dx. \]

Replace \( x = e^y \) (or \( y = \ln x \)) in formula (8) and since \( x = 1 \) to \( \infty \), \( y = 0 \) (\( \ln 1 \)) to \( \infty \) (\( \ln \infty \)).

\[ \int_{1}^{\infty} f(x) x^{-s-1} dx = \int_{0}^{\infty} f(e^y) e^{-sy-y} d(e^y) = \int_{0}^{\infty} f(y) e^{-sy} dy, \]

which is formula (7).

The Bilateral Laplace Transform

Formula (7) is usually called the **Unilateral** Laplace transform since the integral is evaluated from 0 to \( \infty \). The integral below is known as the Bilateral Laplace transform because the integral is taken from \(-\infty\) to \( \infty \),

(9) \[ B\{f\} = \int_{-\infty}^{\infty} f(y) e^{-sy} dy. \]

Now, consider the integral equation

(10) \[ \int_{0}^{\infty} f(x) x^{-s-1} dx, \]

Replace \( x = e^y \) (or \( y = \ln x \)) in formula (4) and since \( x = 0 \) to \( \infty \), \( y = -\infty \) to \( \infty \), thus

\[ \int_{0}^{\infty} f(x) e^{-sx} dx = \int_{-\infty}^{\infty} f(e^y) e^{-sy-y} d(e^y) = \int_{-\infty}^{\infty} f(y) e^{-sy} dy, \]

which is (9).
# Riemann Transform: General Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(s) = R[f(x)] = \int_1^\infty f(x)x^{s-1} , dx )</td>
<td>Definition of Transform</td>
</tr>
<tr>
<td>( f(x) = R^{-1}(F(s)) )</td>
<td>Inverse Transform</td>
</tr>
<tr>
<td>( R[af(x) + bg(x)] = aR[f(x)] + bR[g(x)] )</td>
<td>Linearity</td>
</tr>
<tr>
<td>( R { x^a f(x) } = F(s-a) )</td>
<td>s-Shifting Theorem</td>
</tr>
<tr>
<td>( R^{-1}[F(s-a)] = x^a f(x) )</td>
<td></td>
</tr>
<tr>
<td>( R(f') = (s+1)F(s+1) - f(1) )</td>
<td>Differentiation of Function</td>
</tr>
<tr>
<td>( R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f'(1) )</td>
<td></td>
</tr>
<tr>
<td>( R \left[ \int_1^x f(\tau) , d\tau \right] = \frac{1}{s}F(s-1), )</td>
<td>Integration of Function</td>
</tr>
<tr>
<td>( R[\ln xf(x)] = -F'(s) )</td>
<td>Differentiation of Transform</td>
</tr>
<tr>
<td>( R \left[ \frac{f(x)}{\ln x} \right] = \int_s^\infty F(\sigma) , d\sigma )</td>
<td>Integration of Transform</td>
</tr>
</tbody>
</table>
Table: Some Riemann Transforms

<table>
<thead>
<tr>
<th></th>
<th>$f(x) = R^{-1}{F(s)}$</th>
<th>$F(s) = \int_1^x f(x)x^{-s-1} , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>2</td>
<td>$x$</td>
<td>$\frac{1}{s-1}$</td>
</tr>
<tr>
<td>3</td>
<td>$x^a$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>4</td>
<td>$x^{\alpha i}$</td>
<td>$\frac{1}{s-\alpha i}$</td>
</tr>
<tr>
<td>5</td>
<td>$\cos(\alpha \ln x)$</td>
<td>$\frac{s}{s^2 + \alpha^2}$</td>
</tr>
<tr>
<td>6</td>
<td>$\sin(\alpha \ln x)$</td>
<td>$\frac{\alpha}{s^2 + \alpha^2}$</td>
</tr>
<tr>
<td>7</td>
<td>$\cosh(a \ln x)$</td>
<td>$\frac{s}{s^2 - a^2}$</td>
</tr>
<tr>
<td>8</td>
<td>$\sinh(a \ln x)$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
</tr>
<tr>
<td>9</td>
<td>$x^b \cos(\alpha \ln x)$</td>
<td>$\frac{s - b}{(s-b)^2 + \alpha^2}$</td>
</tr>
<tr>
<td>10</td>
<td>$x^b \sin(\alpha \ln x)$</td>
<td>$\frac{\alpha}{(s-b)^2 + \alpha^2}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{2}{\ln x}</td>
<td>1 - \cos(\alpha \ln x)</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{1}{\ln x}\sin(\alpha \ln x)$</td>
<td>$\arctan\frac{\alpha}{s}$</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{2}{\ln x}</td>
<td>1 - \cosh(a \ln x)</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{1}{\ln x}(x^b - x^a)$</td>
<td>$\ln\left(\frac{s-a}{s-b}\right)$</td>
</tr>
</tbody>
</table>

REFERENCE

Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity*. pp. 5-7.