A PROOF FOR BEAL'S CONJECTURE

Abstract. In the first part of this paper, we show how $a^x - b^y$ can be expressed as a binomial expansion (to an indeterminate power, z). In the second part we will show how this leads to a proof for the Beal Conjecture.

Introduction In 1993, a Texan number theory enthusiast named Andrew Beal conjectured that co-prime bases for the equation $A^x = B^y + C^z$ might be impossible for values of x, y, z greater than 2, where A, B, C are co-prime integers. This is commonly known as Beal's Conjecture (BC)[1] or sometimes the Mauldin Conjecture or the Tijdeman-Zagier Conjecture. It states that if $A^x = B^y + C^z$, where A, B, C, are positive co-prime integers and x, y, z are all positive co-prime integers greater than 2, then A, B, C, must have a common prime factor.
Here, we rearrange the equation as $a^x - b^y = c^z$ without loss of integrity to demonstrate how $a^x - b^y$ can be reconfigured and expressed as a binomial expansion, containing not only the standard factors for a single power but also an additional non-standard factor. We then give a simple proof for Beal's Conjecture.

Definition 0.1. For the equation $a^x - b^y = c^z$, we define a, b, c, as square-free positive integers (of which one at most must be even); and x, y, z are positive integers (of which one at most may be even), and $\gcd(x, y, z) = 1$,

Lemma 0.2. *To demonstrate that $a^x - b^y$ can be expressed as a binomial formula.*

We first observe that by adding $[ab(x^{x-2} - b^{y-2}) - b^y]$ to a^x and b^y respectively, and then rearranging, it is possible to reconfigure the expression such that:

$$a^x - b^y = (a+b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2}). \tag{0.1}$$

Now, since $a^x - b^y = (a+b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2})$, we can repeat the process in the same way by reconfiguring the components $(a^{x-1} - b^{y-1})$ and $(a^{x-2} - b^{y-2})$ respectively, and expanding it as follows:

(0.2)
$$(a+b)[(a+b)(a^{x-2} - b^{y-2}) - ab(a^{x-3} - b^{y-3)})] - ab[(a+b)(a^{x-3} - b^{y-3}) - ab(a^{x-4} - b^{y-4)})]$$

$$= (a+b)^2(a^{x-2} - b^{y-2}) - 2ab(a+b)(a^{x-3} - b^{y-3}) + (ab)^2(a^{x-4} - b^{y-4}). \tag{0.3}$$

Repeating the process for $(a^{x-2} - b^{y-2})$, $(a^{x-3} - b^{y-3})$ and $(a^{x-4} - b^{y-4})$, we get:

(0.4)
$$(a+b)^3(a^{x-3} - b^{y-3}) - 3ab(a+b)^2(a^{x-4} - b^{y-4}) + 3(ab)^2(a+b)(a^{x-5} - b^{y-5}) - (ab)^3(a^{x-6} - b^{y-6}).$$

We can continue to expand $(a^x - b^y)$ *ad infinitum*, and using the binomial formula we can generalise it, for all $z \in \mathbb{Z}$, in a non-standard binomial formula:

$$a^x - b^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k} (-ab)^k (a^{x-z-k} - b^{y-z-k}). \tag{0.5}$$

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[1]See www.bealconjecture.com. Last accessed 14.12.17.

For comparison, the *standard form* of the binomial theorem (also in z) is:

$$(p-q)^z = \sum_{k=0}^{z} \binom{z}{k} p^{z-k}(-q)^k.$$

Remark 0.3. This new non-standard binomial formula in (0.5) is strange in two respects. First, the z-power is indeterminate. That is to say, is value is not determined by $a^x - b^y$, so that regardless of the value we give to z, the value of $(a^x - b^y)$ never changes. It is this property that allows us to compare the two sides of the Beal equation more easily.
Secondly, the first three factors of the sum in (0.5), i.e. $\binom{z}{k}$, $(a+b)^{z-k}$, and $(-ab)^k$ all obviously conform to the conventional forms of a standard binomial formula. But the last factor, $(a^{x-z-k} - b^{y-z-k})$, does not *obviously* conform, but may do. Let us now turn to the main theorem and proof by contradiction.

Theorem 0.4. *To prove that, for the equation $a^x - b^y = c^z$, integer solutions only exist for the values of x or y or $z = 1, 2$, but not for values of $x, y, z > 2$.*

Proof. We first assume that there exists a solution for the equation $a^x - b^y = c^z$ for values of $x, y, z > 2$. So if $c^z = a^x - b^y$ then it follows, from (0.5), that:

$$c^z = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} - b^{y-z-k}). \tag{0.6}$$

Now let $|s| = \frac{c+(ab)|t|}{(a+b)}$ and let $|t| = \frac{c-(a+b)|s|}{(-ab)}$ for all $s, t \in \mathbb{Q}$, such that, for all possible values of c:

$$[(a+b)|s| - ab|t|] = c, \tag{0.7}$$

$$\Rightarrow [(a+b)|s| - ab|t|]^z = c^z. \tag{0.8}$$

Remark 0.5. Note that a) we are using absolute values of s and t in order to preserve the minus sign in (0.8), which is necessary if we are to have comparable forms in (0.10), and b) although $|s|$ and $|t|$ could, in theory, introduce unwelcome fractions, they will soon drop from the proof.

For now, using the standard binomial theorem, it follows from (0.8) that;

$$c^z = \sum_{k=0}^{z} \binom{z}{k} [(a+b)s]^{z-k}[(-ab)t]^k. \tag{0.9}$$

Rearranging this slightly we can say, from (0.6) and (0.9), that:

(0.10)
$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} - b^{y-z-k}) = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k|s|^{z-k}|t|^k.$$

Remark 0.6. According to the normal rules of binomial expansion, the common factors on both sides of the equation (i.e. $\binom{z}{k}$, $(a+b)^{z-k}$ and $(-ab)^k$) will exactly correspond in each and every k^{th} term. So for this equation to have solutions it is also necessary that the remaining factors on both sides, $(a^{x-z-k} - b^{y-z-k})$ and $(|s|^{z-k}|t|^k)$, must also correspond exactly in each counterpart (k^{th}) term, for any given value of z. If it does, then the whole of the left hand side will be a power to z (as we know the right hand side is), and the Beal equation will have solutions.

But if just one term of the corresponding binomials exists where $(a^{x-z-k} - b^{y-z-k})$ does not equal $(|s|^{z-k}|t|^k)$, then not only will the integrity of that particular k^{th} term be compromised as a valid binomial term, but also the whole expression as an expansion of a power to z. In the latter scenario, no solutions will exist.

What we will now proceed to show is that when $z > 2$ an inequality arises in at least one of the corresponding k^{th} terms (for all k terms of the sum *simultaneously* from $k = 0$ to $k = z$), but that when $z = 1, 2$ every corresponding k^{th} term will be equal. Without testing for every value of z one by one *ad infinitum* we will test for all values of $z > 2$ in one go by using the generalised equation:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k = \sum_{k=0}^{z}(a^{x-z-k} - b^{y-z-k}) \tag{0.11}$$

Lemma 0.7. *To prove that when $z > 2$ an inequality arises in at least one of the corresponding k^{th} terms, but that when $z = 1, 2$ every corresponding k^{th} term will be equal.*

Continuing our initial assumption that $c^z = a^x - b^y$ we can now say that:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k = \sum_{k=0}^{z}(a^{x-z-k} - b^{y-z-k}). \tag{0.12}$$

But how can we do this for all values of z and k (*ad infinitum*)? In three steps. First, from the first and last terms (where s and t occur on their own) we will establish the respective values of $|s|, |s|^z, |s|^{z-1}, |t|, |t|^z$, and $|t|^{z-1}$ (in terms of a and b). Secondly we will use these results to evaluate what the *second* and *penultimate* terms are, and compare them with the second and penultimate terms directly derived from $(a^{x-z-k} - b^{y-z-k})$. Thirdly, we will substitute like-terms to reveal the contradictions when they occur. [We will not need to look beyond the second and penultimate terms (even if z is very large) since this is where we find the contradiction in all cases of $z > 2$.]

STEP 1

Using the equation in (0.12), we can establish the respective values of $|s|$, $|s|^z$, $|s|^{z-1}$, $|t|$, $|t|^z$, and $|t|^{z-1}$, using first and last terms (i.e. $k = 0$ and $k = z$). So when $k = 0$, the first term in the binomial series is $|s|^z$, such that:

$$|s|^z = \pm(a^{x-z} - b^{y-z}), \tag{0.13}$$

from which it follows that:

$$|s| = \pm(a^{x-z} - b^{y-z})^{1/z}, \tag{0.14}$$

and

$$|s|^{z-1} = \pm(a^{x-z} - b^{y-z})^{(z-1)/z}. \tag{0.15}$$

Likewise, when $k = z$, the last term in the binomial series is $|t|^z$, such that:

$$|t|^z = \pm(a^{x-2z} - b^{y-2z}), \tag{0.16}$$

from which it follows that:

$$|t| = \pm(a^{x-2z} - b^{y-2z})^{1/z}, \tag{0.17}$$

and

$$|t|^{z-1} = \pm(a^{x-2z} - b^{y-2z})^{(z-1)/z}. \tag{0.18}$$

STEP 2
Using these different values of $|s|$ and $|t|$, we are now in a position to work out what the *second* and *penultimate* terms (in terms of a and b). Thus, from (0.15) and (0.17), it follows that the *second* term, $|s|^{z-1}|t|$, is:

$$\pm(a^{x-z} - b^{y-z})^{(z-1)/z}(a^{x-2z} - b^{y-2z})^{1/z}. \tag{0.19}$$

And since we know, from the right hand side of the equation in (0.12), that the second term in the binomial expansion is $\pm(a^{x-z-1} - b^{y-z-1})$, i.e. when $k = 1$, it follows that:

$$\pm(a^{x-z} - b^{y-z})^{(z-1)/z}(a^{x-2z} - b^{y-2z})^{1/z} = \pm(a^{x-z-1} - b^{y-z-1}). \tag{0.20}$$

Dividing both sides by $\pm(a^{x-z} - b^{y-z})^{(z-2)/z}$ we get:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{1/z} = \frac{\pm(a^{x-z-1} - b^{y-z-1})}{\pm(a^{x-z} - b^{y-z})^{(z-2)/z}}. \tag{0.21}$$

It follows from (0.14) and (0.18) that the *penultimate* term, $|s||t|^{z-1}$, is:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{(z-1)/z}. \tag{0.22}$$

And since we know, from the right hand side of the equation in (0.12), that the penultimate term in the binomial expansion is $\pm(a^{x-2z+1} - b^{y-2z+1})$, i.e. when $k = z - 1$, it follows that:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{(z-1)/z} = \pm(a^{x-2z+1} - b^{y-2z+1}). \tag{0.23}$$

Dividing both sides by $(a^{x-2z} - b^{y-2z})^{(z-2)/z}$ we get:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{1/z} = \frac{\pm(a^{x-2z+1} - b^{y-2z+1})}{\pm(a^{x-2z} - b^{y-2z})^{(z-2)/z}}. \tag{0.24}$$

STEP 3
Thirdly, we are in a position to substitute like-terms. For the left hand sides of the equations in (0.21) and (0.24) are exactly the same. Therefore by substituting like-terms we get:

$$\frac{\pm(a^{x-z-1} - b^{y-z-1})}{\pm(a^{x-z} - b^{y-z})^{(z-2)/z}} = \frac{\pm(a^{x-2z+1} - b^{y-2z+1})}{\pm(a^{x-2z} - b^{y-2z})^{(z-2)/z}} \tag{0.25}$$

We raise both sides by the power of z and rearrange to get:

$$\pm\left(\frac{a^{x-z-1} - b^{y-z-1}}{a^{x-2z+1} - b^{y-2z+1}}\right)^{z} = \pm\left(\frac{a^{x-z} - b^{y-z}}{a^{x-2z} - b^{y-2z}}\right)^{(z-2)} \tag{0.26}$$

We will return shortly to the case of $z = 1, 2$, but for now (still assuming that $x, y, z > 2$) we can say that solutions will exist either a) if the large bracketed factors on each side of the equation in (0.26) have a value of 1 (since the main outer exponents are not equal), or b) if the numerators in (0.26) are equal *and* simultaneously if the denominators are equal, *and* in both cases, of course, if the polarity of signs in front of the brackets are the same on both sides. Taking these two options in turn:

a) since $(a^{x-z-1} - b^{y-z-1}) \neq (a^{x-2z+1} - b^{y-2z+1})$, and $(a^{x-2z} - b^{y-2z}) \neq (a^{x-z} - b^{y-z})$, neither side in (0.26) has a value of 1, eliminating this option;

b) *even without* exponents $(a^{x-2z+1} - b^{y-2z+1})$ is greater than $(a^{x-2z} - b^{y-2z})$; but with a *higher* exponent, z (i.e. which is greater than $z - 2$), the inequality is even greater. So it follows that $(a^{x-2z+1} - b^{y-2z+1})^z \neq (a^{x-2z} - b^{y-2z})^{(z-2)}$. Having now eliminated the options it follows that, for all values of $x, y, z > 2$:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k \neq \sum_{k=0}^{z} (a^{x-z-k} - b^{y-z-k}). \tag{0.27}$$

But this contradicts our second equation in (0.12), thus proving Lemma 0.7. And so our assumption that for any value of $x, y, z > 2$ solutions exist for the equation $c^z = a^x - b^y$ is also false. Thus, Beal's Conjecture is true. □

Remark 0.8. We have now proved BC, but the question remains about the cases of $z = 1, 2$. Well, these cases resolve neatly, if unpredictably. From (0.26), when $z = 1$ (and again when there is equal polarity of pre-bracket signs) it follows that:

$$\left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right)^1 = \left(\frac{a^{x-1} - b^{y-1}}{a^{x-2} - b^{y-2}}\right)^{-1} \tag{0.28}$$

$$\Rightarrow 1 = 1. \tag{0.29}$$

Thus, when the signs are equal on both sides, there is no contradiction. And again from (0.26), when $z = 2$ (and there is equal polarity of pre-bracket signs), it follows that:

$$\left(\frac{a^{x-3} - b^{y-3}}{a^{x-3} - b^{y-3}}\right)^2 = \left(\frac{a^{x-2} - b^{y-2}}{a^{x-4} - b^{y-4}}\right)^0 \tag{0.30}$$

$$\Rightarrow 1 = 1. \tag{0.31}$$

Again, no contradiction. So in both cases, when $z = 1$ and when $z = 2$, the standard rules of binomial expansion can be applied to our non-standard binomial expression without contradiction such that $(a^{x-z-k} - b^{y-z-k})$ is equal to $|s|^{z-k}|t|^k$, and therefore that in these cases solutions to the original equation exist.

Finally, it is worth mentioning the obvious point that we can apply the same method to Fermat's Last Theorem with the same result.

References

1. Peter Schorer, *Is There a "Simple" Proof of Fermat's Last Theorem? Part (1) Introduction and Several New Approaches*, 2014, www.occampress.com/fermat.pdf last accessed 14.12.17.
2. Andrew Wiles, *Modular Elliptic Curves and Fermat's Last Theorem*, Ann. of Math (2), Vol. 141, No.3 (May, 1995), 443-551.
3. www.bealconjecture.com. Last accessed 14.12.17.

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