An alternate view of the elliptic property of a family of projectile paths

Radhakrishnamurty Padyala
282, DMLO, Yelahanka, Bengaluru -560064, India
Email: padyala1941@yahoo.com

Abstract

Fernandez-Chapou and coworkers analyzed projectile trajectories and showed an elliptic property hidden in them. They pointed out and demonstrated a little known property of these projectiles, namely, the vertices of these projectiles lie on an ellipse. We present in this article an alternate view point of observing this ellipse. We emphasize, that we don't use vectors or calculus or Newton's laws in our analysis. We follow the traditions of Galileo – we use geometry principles. We consider here the semi parabolic paths of the projectiles.

Ellipse of a family of projectiles shot from a point with common speed and different angles

Fernandez-Chapou and coworkers analyzed projectile trajectories and showed an elliptic property hidden in them[1]. For that analysis, they considered projectiles shot from a point with a common value of speed and different angles of projection. Such projectile paths exhibit some interesting characteristics. For example, pairs of projectiles with complimentary angles of projection have common values of ranges. This is a well-known property of ideal projectile motion. However, they pointed out and demonstrated a little known property of these projectiles, namely, the vertices of these projectiles lie on an ellipse. It is the ellipse of the family of projectiles shot from a point with common speed and different angles.

We present in this article an alternate view point of observing this ellipse. We emphasize, that we don't use vectors or calculus or Newton's laws in our analysis. We follow the traditions of Galileo – we use geometry principles. We consider here the semi parabolic paths of the projectiles. Let us consider a circle of arbitrary radius, 'a' in a vertical plane. We draw vertical and horizontal diameters A_1B and CD respectively. From points A_i on the circle in the first quadrant, we draw vertical lines A_iO_i to CD (see Fig.1). We place plane mirrors inclined at 45 degrees to the horizontal, at m_i, the mid points[♠] of the lines A_iO_i. We let particles fall from rest vertically down from points A_i. At m_i the particles hit the mirrors and get reflected to move along the horizontal direction with the speed v_i, acquired in falling through A_im_i. (The speed v_i is equal to half the speed in falling through A_iO_i. It is also the average speed in falling through A_iO_i). As they do so, the particles are subjected to uniform acceleration in the vertically down-ward direction. Thus the particles are subjected to a uniform motion of constant speed along the horizontal direction and uniformly accelerated motion along the vertically down-ward direction. As a result of these simultaneous motions in orthogonal directions, the particles execute a semi-parabolic motion in the vertical plane. We can get the corresponding paths in the second quadrant from symmetry considerations (here, by reflection). The starting points of these parabolic motions (positions of mirrors) are the vertices of the parabolic paths. The locus of these vertices gives the upper half of an ellipse.

♠ Choosing mid points has significance. The speed attained at mid-point of A_iO_i represents the time average speed for the fall over the distance A_iO_i.

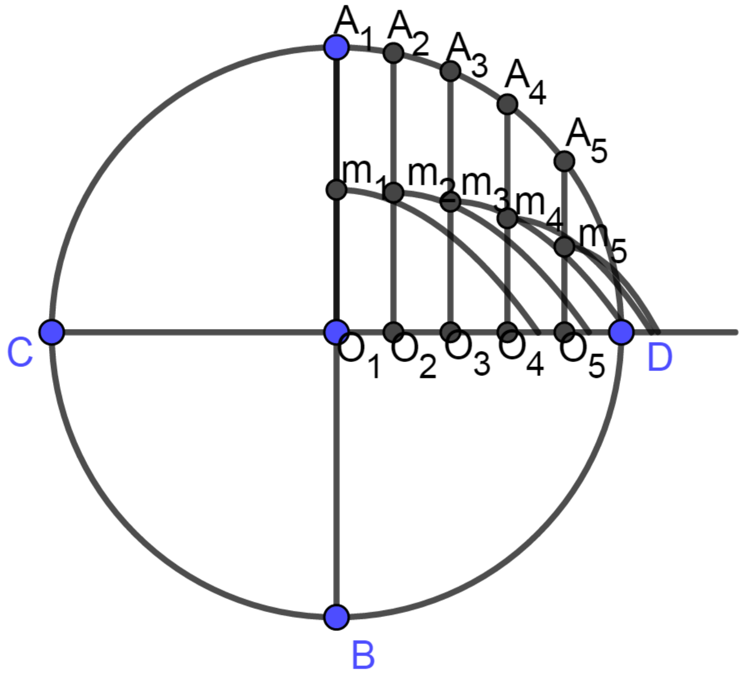

Fig. 1. Mirrors are located at mi . Particles dropped from Ai hit the mirrors, get reflected and follow the parabolic paths there after.

A reflection of this in the horizontal CD gives the lower half of the ellipse. This narration is depicted in Fig. 2. It is easy to show that the location of mirrors which correspond to the vertices of the parabolas lie on an ellipse.

Let the equation of the circle be,

$$x^2 + y^2 = a^2 \tag{1}$$

Where x, y are the coordinates of any point on the circle with its center at the origin. Let the coordinates of the locations of the mirrors be x' and y'. The pairs x, y and x', y' are related as,

$$x' = x \text{ and } y' = \frac{1}{2}y \tag{2}$$

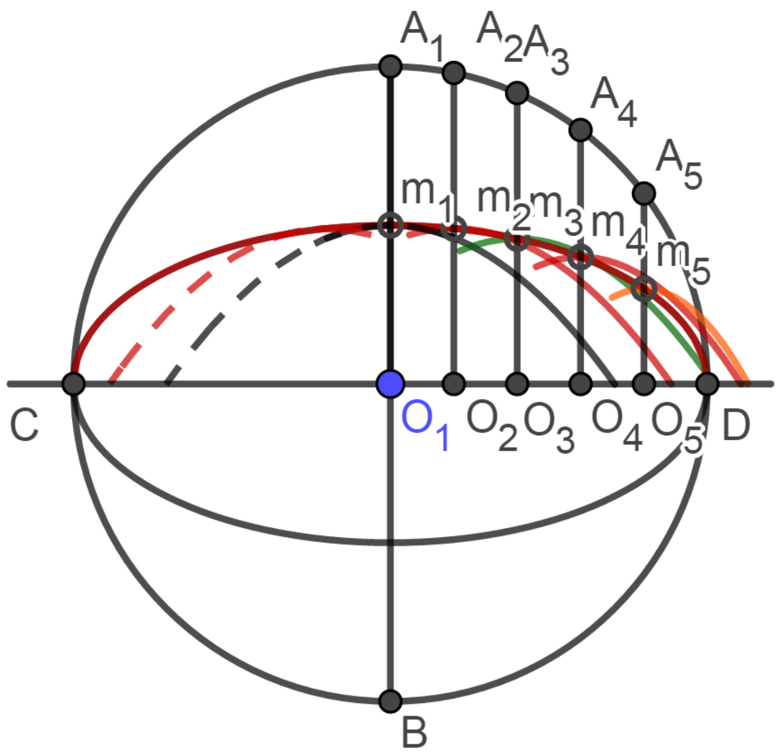

Fig. 2. Ellipse passing through the apexes of the parabolas

From (1) and (2) we get

$$x'^2 + (2y')^2 = a^2 \tag{3}$$

Dividing both sides by a^2, we get,

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{\left(\frac{a}{2}\right)}\right)^2 = 1 \tag{4}$$

Equation (4) is in the form of

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \tag{5}$$

Equation (5) is the equation of an ellipse centered at the origin and having an eccentricity of √3/2.

We show below that this ellipse is the same as the one shown in reference 1.

Let particles fall from rest from the upper end A of the vertical diameter AB of the circle centered at the origin, with uniform acceleration g in the vertically down-ward direction. When they reach the lower end B, they would have acquired a speed v_B. We locate a plane mirror at B. The inclination of the mirror to the horizontal can be adjusted to any desired value. When the particles strike the mirror on their way from A, they get reflected so as to move along a plane inclined to the horizontal at the desired angle α_j. Let this plane be represented by the chord of the circle drawn from B (for example BA'). The particles reach the other end of the chords in the same interval of time they took in falling from A to B along the vertical. (see Fig.3) This follows from the law of chords of Galileo[2].

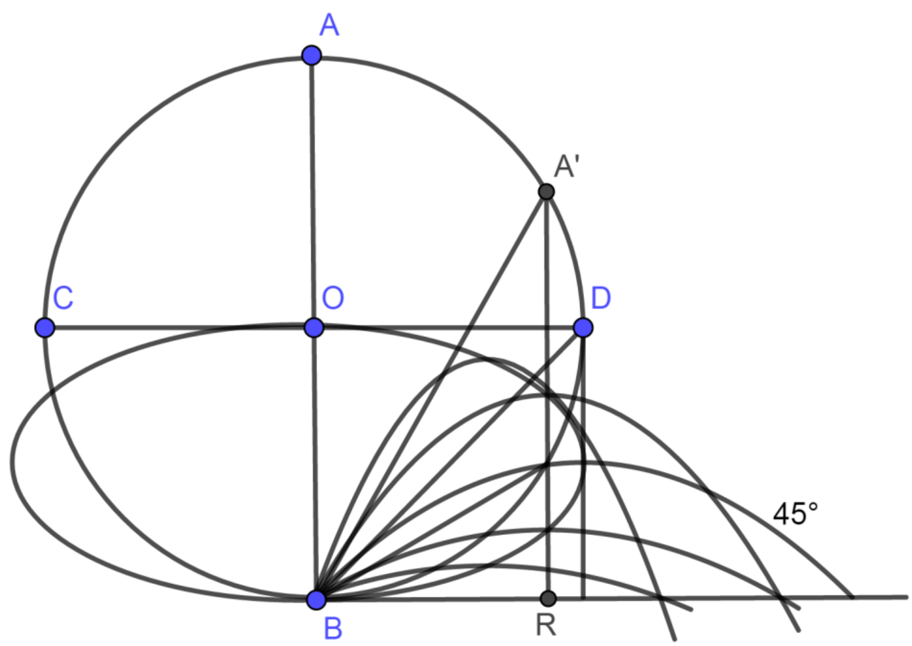

Fig. 3 The family of parabolic paths of projectiles shot from point B with constant speed v_B at different angles from the horizontal. The ellipse that passes through the epixes of the parabolic paths is shown in the figure. This corresponds to the ellipse shown in reference 1.

Instead of travelling along the inclined planes after reflection at B, if the particles move as projectiles shot from B with speed v_B at an angle α_j with constant acceleration in the vertically down-ward direction, they trace parabolic paths. As the particle that traverses along the inclined plane (chord from B) reaches the other end of the chord of the circle, the particle moving along the parabolic path reaches the vertex of the parabola. Thus the travel times of particles falling vertically from A to B, the travel times of particles travelling the lengths of the chords from B, and the travel times of the particles travelling from B along the parabolic paths to the apexes, are all equal. This is an important result that should be noted. By symmetry it follows that the particle takes an equal time to reach the ground from the apex as it took to reach the apex from the instant of launch from B. The horizontal component of the chord corresponds to half the range of the projectile launched with speed v_B and angle of projection α_j. It is clear from Fig. 3 that the range is maximum for the projectile with angle of projection of 45 degrees, and equals the radius of the circle.

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References

1. J. L. Fernandez-Capou, A. L. Salas-Brito, and C. A. Vargas, "*An elliptic property of parabolic trajectories*," Am. J. Phys. ***72***, 1109 (2004).
2. David K. Hill, *"Galileo's work on 116v: A New Analysis,"* Isis ***77***, 283-291 (1986).