γ is Irrational

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Abstract

We introduce an unaccustomed number system, $\mathbb{H}^{\pm}$, and show how it can be used to prove γ is irrational. This number system consists of plus and minus multiplies of the terms of the harmonic series. Using some properties of ln, this system can depict the harmonic series and $\lim_{n\to\infty} \ln n$ at the same time, giving γ as an infinite decimal. The harmonic series converges to infinity so negative terms are forced. As all rationals can be given in $\mathbb{H}^{\pm}$ without negative terms, it follows that γ must be irrational.

1 Introduction

We are used to the positional number system. Viewing such numbers as symbols, things that represent something, .111, in base 10, really has three different number symbols: 1/10, 1/100, and 1/1000. In other bases we have other symbols, but still a fixed set of symbols is used within common positional notation systems: there are in this sense a finite number of symbols. We relax these conventions and form a new number base that has an infinite number of symbols and allows for symbols that depict negative fractions to be summed as well. This number base can depict all rational numbers in $(0, 1)$ a certain way; it also can depict γ. If γ can't be depicted in one of the ways rational numbers can, it follows that it must be irrational. Proving γ is irrational is an open number theory problem [1, p. 52].

2 γ

The harmonic series is defined by

$$h = \sum_{k=1}^{\infty} \frac{1}{k}.$$

We can use its terms to suggest a set of symbols that define a set of numbers in all possible natural number bases. So in first position, let .1 represent $1/2$ and let .01 be $-1/2$. In position three let .001 be $1/3$, .002 be $2/3$ and so forth: $-1/4 = 0.00_2 00_3 01_4$, where the subscripts suggest the pattern. Each odd and even position has a finite set of symbols given by $\pm 1/n \cdots \pm (n-1)/n$; we allow the first position to range from $3/2 \cdots - 3/2$ to accomodate the unit in h. With this notational system, using the usual overbar to designate repetition of a number pattern, the harmonic series is given by $h = .3\overline{01}$. Any fraction in $(0, 1)$ can be depicted in a finite number of positive symbols: for example, $1/2 + 1/3 = 5/6$ is given by .101. We designate this system as $\mathbb{H}^{\pm}$.

The mathematical constant γ is defined by

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n. \tag{1}$$

That is the harmonic series minus an evolving set of values.

Within $\mathbb{H}^{\pm}$ we can subtract a number provided that it is expressible as sums of fractions and multiplies of fractions within certain limits. We can recast the ln tail of γ as an element in $\mathbb{H}^{\pm}$. The ln function has a power series expansion [2, p. 621] as follows

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

for $|x| < 1$. For the natural number arguments in γ's definition we can use

$$x = \frac{n-1}{n},$$

as

$$
\begin{aligned}
&\ln\left(1-\frac{n-1}{n}\right)\\
&=\ln\left(\frac{n-(n-1)}{n}\right)\\
&=\ln\frac{1}{n}\\
&=-\ln n.
\end{aligned}
$$

Using the power series expansion we then have

$$-\ln n=\ln(1-\frac{n-1}{n})=-\sum_{k=1}^{\infty}\left(\frac{n-1}{n}\right)^k\frac{1}{k}.$$

This is the right form as each term

$$\left(\frac{n-1}{n}\right)^k\frac{1}{k} \tag{2}$$

is less than one and is a fraction. Stop. We have a picture using (2): each of the terms fits into a slot in the negative position. As n grows the harmonic is "eaten into" until it eventually converges to γ.

Our goal is to depict γ in $\mathbb{H}^{\pm}$. If we use a subsequence of the natural numbers that goes to infinity in (1) it will converge as well to γ. We choose

$$P_n=\prod_{k=1}^{n}p_1\dots p_k,$$

that is the growing product of primes. Now $\ln P_n=\ln p_1+\dots\ln p_n$ and we have

$$\lim_{n\to\infty}-\ln P_n=\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{p_r-1}{p_r}\right)^k\frac{1}{k}.$$

As each of the denominators involves a different prime this means that the position in $\mathbb{H}^{\pm}$ is different for each term; there is no overlapping. It can be depicted in $\mathbb{H}^{\pm}$. As this negative part is not positive it does not interfere with anything in the harmonic part. This means $\gamma\in\mathbb{H}^{\pm}$. The tail we note may be differently designated but the head, h, is fixed.

The logic that makes γ irrational is as follows. In $\mathbb{H}^{\pm}$ all rational numbers in $(0, 1)$ can be expressed as the finite sum of positive fractions. We also have that $\gamma \in \mathbb{H}^{\pm}$ and we observe that γ can not be expressed in $\mathbb{H}^{\pm}$ without negative fractions. It follows that γ can't be rational.

References

[1] J. Havil, *Gamma*, Princeton University Press, Princeton, NJ, 2003.

[2] J. Stewart, *Calculus: Early Vectors*, Brooks/Cole, Pacific Grove, CA, 1999.