Research Article

On Pseudohyperbolical Smarandache Curves in Minkowski 3-Space

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In the theory of curves in the Euclidean and Minkowski spaces, one of the interesting problems is the problem of characterization of a regular curve. In the solution of the problem, the curvature functions \( \kappa \) and \( \tau \) of a regular curve have an effective role. It is known that we can determine the shape and size of a regular curve by using its curvatures \( \kappa \) and \( \tau \). Another approach to the solution of the problem is to consider the relationship between the corresponding Frenet vectors of two curves. For instance, Bertrand curves and Mannheim curves arise from this relationship. Another example is the Smarandache curves. They are the objects of Smarandache geometry, that is, a geometry which has at least one Smarandachely denied axiom [1]. An axiom is said to be Smarandachely denied if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes.

If the position vector of a regular curve \( \alpha \) is composed by the Frenet frame vectors of another regular curve \( \beta \), then the curve \( \alpha \) is called a Smarandache curve [2]. Special Smarandache curves in Euclidean and Minkowski spaces are studied by some authors [3–8]. The curves lying on a pseudohyperbolic space \( H^2_0 \) in Minkowski 3-space \( E^3_1 \) are characterized in [9].

In this paper, we define pseudohyperbolical Smarandache curves according to the Sabban frame in Minkowski 3-space. We obtain the geodesic curvatures and the expression for the Sabban frame vectors of special pseudohyperbolic Smarandache curves. Finally, we give some examples of such curves.

2. Basic Concepts

The Minkowski 3-space \( \mathbb{R}^3_1 \) is the Euclidean 3-space \( \mathbb{R}^3 \) provided with the standard flat metric given by

\[
\langle \mathbf{x}, \mathbf{y} \rangle = -dx_1^2 + dx_2^2 + dx_3^2,
\]

where \( (x_1, x_2, x_3) \) is a rectangular Cartesian coordinate system of \( \mathbb{R}^3_1 \). Since \( g \) is an indefinite metric, recall that a nonzero vector \( \mathbf{x} \in \mathbb{R}^3_1 \) can have one of the three Lorentzian causal characters: it can be spacelike if \( \langle \mathbf{x}, \mathbf{x} \rangle > 0 \), timelike if \( \langle \mathbf{x}, \mathbf{x} \rangle < 0 \), and null (lightlike) if \( \langle \mathbf{x}, \mathbf{x} \rangle = 0 \). In particular, the norm (length) of a vector \( \mathbf{x} \in \mathbb{R}^3_1 \) is given by \( \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \) and
two vectors \( \vec{x} \) and \( \vec{y} \) are said to be orthogonal if \( \langle \vec{x}, \vec{y} \rangle = 0 \). Next, recall that an arbitrary curve \( \alpha = \alpha(s) \) in \( \mathbb{E}_2^3 \) can locally be spacelike, timelike, or null (lightlike) if all of its velocity vectors \( \alpha'(s) \) are, respectively, spacelike, timelike, or null (lightlike) for every \( s \in I \) [10]. If \( ||\alpha'(s)|| \neq 0 \) for every \( s \in I \), then \( \alpha \) is a regular curve in \( \mathbb{R}^3_1 \). A spacelike (timelike) regular curve \( \alpha \) is parameterized by pseudo-arclength parameter \( s \) which is given by \( \alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) be a regular unit speed curve lying fully in \( \mathbb{H}^2_0 \) with the Sabban frame \( \{\alpha, T, \xi\} \) and \( (\beta, T, \xi, \beta) \) is the moving Sabban frames of these curves, respectively. Then we have the following definitions of pseudohyperbolical Smarandache curves.

**Definition 3.** Let \( \alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) be a regular unit speed curve lying fully in \( \mathbb{H}^2_0 \). Then \( \alpha T \xi \)-pseudohyperbolical Smarandache curve \( \beta : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) of \( \alpha \) is defined by

\[
\beta(s^*(s)) = \frac{1}{\sqrt{2}} (a\alpha(s) + b\xi(s)),
\]

where \( a, b \in \mathbb{R}_0 \) and \( a^2 - b^2 = 2 \).

**Definition 4.** Let \( \alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) be a regular unit speed curve lying fully in \( \mathbb{H}^2_0 \). Then \( \alpha T \xi \)-pseudohyperbolical Smarandache curve \( \beta : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) of \( \alpha \) is defined by

\[
\beta(s^*(s)) = \frac{1}{\sqrt{2}} (a\alpha(s) + bT(s)),
\]

where \( a, b \in \mathbb{R}_0 \) and \( a^2 - b^2 = 2 \).

**Theorem 6.** Let \( \alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) be a regular unit speed curve lying fully in \( \mathbb{H}^2_0 \). Then \( \alpha T \xi \)-pseudohyperbolical Smarandache curve \( \beta : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) of \( \alpha \) does not exist.

**Proof.** Assume that there exists \( \alpha T \xi \)-pseudohyperbolical Smarandache curve \( \alpha \). Then it can be written as

\[
\beta(s^*(s)) = \frac{1}{\sqrt{2}} (a\alpha(s) + b\xi(s)),
\]

where \( a^2 + b^2 = -2 \), which is a contradiction.

In the theorems which follow, we obtain Sabban frame \( \{\beta, T, \xi, \beta\} \) and geodesic curvature \( k_\eta \) of pseudohyperbolical Smarandache curve \( \beta \).

**Theorem 7.** Let \( \alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2_0 \) be a regular unit speed curve lying fully in \( \mathbb{H}^2_0 \) with the Sabban frame \( \{\alpha, T, \xi\} \) and
the geodesic curvature $k_g$. If $\beta : I \subset \mathbb{R} \mapsto H_0^2$ is a $\alpha \xi$-pseudohyperbolical Smarandache curve of $\alpha$, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} =
\begin{bmatrix}
\alpha \\
T \\
\xi
\end{bmatrix},
$$

(10)

and the corresponding geodesic curvature $k_g^\beta$ reads

$$
k_g^\beta = \frac{ak_g - b}{|a - bk_g|}, \quad a, b \in \mathbb{R}_0^+, \quad \text{where } a^2 - b^2 = 2 \text{ and } e = \pm 1.
$$

Proof. Differentiating (6) with respect to $s$ and using (4), we obtain

$$
\beta'(s) = \frac{d\beta}{ds} = \frac{a - bk_g}{\sqrt{2}} T,
$$

(12)

$$
T_\beta \frac{ds}{ds} = \frac{a - bk_g}{\sqrt{2}} T,
$$

where

$$
\frac{ds}{ds} = \frac{|a - bk_g|}{\sqrt{2}}.
$$

(13)

Therefore, the unit spacelike tangent vector of the curve $\beta$ is given by

$$
T_\beta = \varepsilon T,
$$

(14)

where $\varepsilon = +1$ if $a - bk_g > 0$ for all $s$ and $\varepsilon = -1$ if $a - bk_g < 0$ for all $s$.

Differentiating (14) with respect to $s$, we find

$$
\frac{dT_\beta}{ds} \frac{ds}{ds} = e \left( \alpha + k_g \xi \right),
$$

(15)

and from (13) and (15) we get

$$
T_\beta' = \frac{\sqrt{2}e}{|a - bk_g|} \left( \alpha + k_g \xi \right).
$$

(16)

On the other hand, from (6) and (14) it can be easily seen that

$$
\xi_\beta = \beta \times T_\beta
$$

(17)

$$
= \varepsilon \frac{b}{\sqrt{2}} \alpha + \varepsilon \frac{a}{\sqrt{2}} \xi
$$

is a unit spacelike vector.

Consequently, the geodesic curvature $k_g^\beta$ of the curve $\beta = \beta(s^\star)$ is given by

$$
k_g^\beta = \det \left( \beta_\beta, T_\beta, T_\beta' \right)
$$

(18)

$$
= \frac{ak_g - b}{|a - bk_g|}.
$$

Theorem 8. Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in $H_0^2$ with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature $k_g$. If $\beta : I \subset \mathbb{R} \mapsto H_0^2$ is $\alpha \xi$-pseudohyperbolical Smarandache curve of $\alpha$, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} =
\begin{bmatrix}
\frac{a}{\sqrt{2}} \\
\frac{b}{\sqrt{2}} \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{\sqrt{2} + (bk_g)^2}{2} \\
\frac{2 + (bk_g)^2}{2} \\
\frac{\sqrt{2}}{2}
\end{bmatrix}
\begin{bmatrix}
a \\
bk_g
\end{bmatrix},
$$

(19)

and the corresponding geodesic curvature $k_g^\beta$ reads

$$
k_g^\beta = \frac{b^2k_g e_1 - abk_g e_2 + 2e_3}{(2 + b^2k_g^2)^{3/2}}, \quad a, b \in \mathbb{R}_0^+, \quad \text{where}
$$

$$
e_1 = -b^3k_g k'g + a \left( 2 + (bk_g)^2 \right),
$$

(21)

$$
e_2 = -ab^2 k_g k'g + (b - bk_g^2) \left( 2 + (bk_g)^2 \right),
$$

$$
e_3 = -b^3k_g^2 k'g + (ak_g + bk_g'g) \left( 2 + (bk_g)^2 \right),
$$

and $a^2 - b^2 = 2$.

Proof. Differentiating (7) with respect to $s$ and using (4), we obtain

$$
\beta'(s) = \frac{d\beta}{ds} \frac{ds}{ds} = \frac{1}{\sqrt{2}} \left( b\alpha + aT + bk_g \xi \right),
$$

(22)

$$
T_\beta \frac{ds}{ds} = \frac{1}{\sqrt{2}} \left( b\alpha + aT + bk_g \xi \right),
$$

where

$$
\frac{ds}{ds} = \sqrt{\frac{2 + (bk_g)^2}{2}}.
$$

(23)

Therefore, the unit spacelike tangent vector of the curve $\beta$ is given by

$$
T_\beta = \frac{1}{\sqrt{2 + (bk_g)^2}} \left( b\alpha + aT + bk_g \xi \right).
$$

(24)
Differentiating (24) with respect to $s$, it follows that

$$\frac{dT_\beta}{ds} = \frac{1}{(2 + (bk_g)^2)^{3/2}} (e_1 \alpha + e_2 T + e_3 \xi).$$

(25)

From (4) and (23), we get

$$T'_\beta = \frac{\sqrt{2}}{(2 + (bk_g)^2)^{3/2}} (e_1 \alpha + e_2 T + e_3 \xi),$$

(26)

where

$$e_1 = -b^2 k_g k'_g + a (2 + (bk_g)^2),$$

$$e_2 = ab^2 k_g k'_g + (b - bk_g^2)(2 + (bk_g)^2),$$

$$e_3 = -b^3 k'_g k''_g + (ak_g + bk_g') (2 + (bk_g)^2).$$

On the other hand, from (7) and (24), it can be easily seen that

$$\xi_\beta = \beta \times T_\beta$$

$$= -\frac{b^2 k_g}{\sqrt{4 + 2(bk_g)^2}} \alpha - \frac{abk_g}{\sqrt{4 + 2(bk_g)^2}} T + \frac{\sqrt{2}}{2 + (bk_g)^2} \xi.$$

(28)

Hence $\xi_\beta$ is a unit spacelike vector.

Therefore, the geodesic curvature $k^\beta_g$ of the curve $\beta = \beta(s^*)$ is given by

$$k^\beta_g = \det (\beta, T_\beta, T'_\beta)$$

$$= \frac{b^2 k_g e_1 - abk_g e_2 + 2e_3}{(2 + b^2 k_g^2)^{3/2}}.$$

(29)

**Theorem 9.** Let $\alpha : I \subset \mathbb{R} \rightarrow H^2_0$ be a regular unit speed curve lying fully in $H^2_0$ with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature $k^\beta_g$. If $\beta : I \subset \mathbb{R} \rightarrow H^2_0$ is $aT\xi$-pseudohyperbolic Smarandache curve of $\alpha$, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} =
\begin{bmatrix}
a \\
b \\
a - ck_g
\end{bmatrix}
\begin{bmatrix}
\sqrt{3} \\
a - ck_g \\
3 + c^2 - ack_g
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
bc - abk_g
\end{bmatrix}
\begin{bmatrix}
b \\
\sqrt{3} \\
3 + c^2 - acc_g
\end{bmatrix}
\begin{bmatrix}
c
\sqrt{3}
\end{bmatrix},$$

(30)

and the corresponding geodesic curvature $k^\beta_g$ reads

$$k^\beta_g = \left( - (ac - (a^2 - 3) k_g) e_1 
+ (bc - abk_g) e_2 + (3 + c^2 - acc_g) e_3 \right) \times \left( \left( (ak_g - c)^2 - 3k_g^2 + 3 \right)^{3/2} \right)^{-1},$$

(31)

where

$$e_1 = -(ak_g - c)(ak'_g - 3k'_g k_g)(a - ck_g),$$

$$e_2 = -(ak_g - c)(ak'_g - 3k'_g k_g)(a - ck_g)$$

$$+ \left( (ak_g - c)^2 - 3k_g^2 + 3 \right)(b - cbb_g - b^2 k_g^2),$$

$$e_3 = -(ak_g - c)(ak'_g - 3k'_g k_g)bk_g$$

$$+ \left( (ak_g - c)^2 - 3k_g^2 + 3 \right)((a - ck_g)k_g + bkg'_g).$$

(32)

and $a^2 - b^2 - c^2 = 3$.

**Proof.** Differentiating (8) with respect to $s$ and by using (4), we find

$$\beta'(s) = \frac{d\beta}{ds^*} = \frac{1}{\sqrt{3}} \left( b\alpha + (a - ck_g) T + bk_g \xi \right).$$

(33)
and thus
\[ T_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{3}} \left( b\alpha + (a - ck_g) T + bk_g \xi \right), \]  
(34)

where
\[ \frac{ds^*}{ds} = \sqrt{\frac{(ak_g - c)^2 - 3k_g^2 + 3}{3}}. \]  
(35)

The geodesic curvature \( k_g \neq (ac \pm \sqrt{a^2c^2 - (a^2 - 3)(c^2 + 3)})/(a^2 - 3) \) for all \( s \), since \( \beta = \beta(s^*) \) is a unit speed regular curve in \( \mathbb{R}^3 \).

Therefore, the unit spacelike tangent vector of the curve \( \beta \) is given by
\[ T_\beta = \frac{1}{\sqrt{\frac{(ak_g - c)^2 - 3k_g^2 + 3}{3}}} \left( b\alpha + (a - ck_g) T + bk_g \xi \right). \]  
(36)

Differentiating (36) with respect to \( s \) and from (4) and (35), it follows that
\[ T_\beta'' = \frac{\sqrt{3}}{\left( \frac{(ak_g - c)^2 - 3k_g^2 + 3}{3} \right)^2} \left( e_1\alpha + e_2 T + e_3 \xi \right), \]  
(37)

where
\[ e_1 = -\left( (ak_g - c)ak_g' - 3k_g'k_g \right) b \]
\[ + \left( (ak_g - c)^2 - 3k_g^2 + 3 \right) (a - ck_g), \]
\[ e_2 = -\left( (ak_g - c)ak_g' - 3k_g'k_g \right) (a - ck_g) \]
\[ + \left( (ak_g - c)^2 - 3k_g^2 + 3 \right) (b - ck_g' - bk_g^2), \]
\[ e_3 = -\left( (ak_g - c)ak_g' - 3k_g'k_g \right) bk_g \]
\[ + \left( (ak_g - c)^2 - 3k_g^2 + 3 \right) ((a - ck_g) k_g + bk_g'). \]

On the other hand, from (8) and (36) it can be easily seen that
\[ \xi_\beta = \beta \times T_\beta \]
\[ = \frac{-(a^2 - 3)k_g + ac}{\sqrt{3(ak_g - c)^2 - 9k_g^2 + 9}} \alpha \]
\[ + \frac{bc - abk_g}{\sqrt{3(ak_g - c)^2 - 9k_g^2 + 9}} \]
\[ + \frac{3 + c^2 - ack_g}{\sqrt{3(ak_g - c)^2 - 9k_g^2 + 9}} \xi. \]  
(39)

Hence \( \xi_\beta \) is a unit spacelike vector.

Therefore, the geodesic curvature \( k_g^\beta \) of the curve \( \beta = \beta(s^*) \) is given by
\[ k_g^\beta = \det \left( \beta, T_\beta, T_\beta'' \right) \]
\[ = \left( - (ac - (a^2 - 3)k_g) e_1 + (bc - abk_g) e_2 \right. \]
\[ + \left( 3 + c^2 - ack_g \right) e_3 \]
\[ \times \left( \left( (ak_g - c)^2 - 3k_g^2 + 3 \right)^{5/2} \right)^{-1}. \]  
(40)

\textbf{Corollary 10.} If \( \alpha : I \subset \mathbb{R} \mapsto H_0^2 \) is a geodesic curve on \( H_0^2 \) in Minkowski 3-space \( E_3^1 \), then

1. \( \alpha T^\perp \)-pseudohyperbolic Smarandache curve is also geodesic on \( H_0^2 \);

2. \( \alpha \xi^\perp \)-pseudohyperbolic and \( \alpha T_\xi^\perp \)-pseudohyperbolic Smarandache curves have constant geodesic curvatures on \( H_0^2 \).

\[ \alpha \]

\[ \text{Example 1.} \] Let \( \alpha \) be a unit speed curve lying in pseudohyperbolic space \( H_0^2(1) \) in the Minkowski 3-space \( E_3^1 \) with parameter equation (see Figure 1)
\[ \alpha(s) = \left( \frac{s^2}{2} + 1, \frac{s^2}{2}, s \right). \]  
(41)
The orthonormal Sabban frame \(\{\alpha(s), T(s), \xi(s)\}\) along the curve \(\alpha\) is given by
\[
\alpha(s) = \left(\frac{s^2}{2} + 1, \frac{s^2}{2}, s\right),
\]
\[
T(s) = \alpha'(s) = (s, s, 1),
\]
\[
\xi(s) = \alpha(s) \times T(s) = \left(\frac{s^2}{2}, \frac{s^2}{2} - 1, s\right).
\]  

In particular, the geodesic curvature \(k_\alpha\) of the curve \(\alpha\) has the form
\[
k_\alpha(s) = -1.
\]  

**Case 1.** If we take \(a = 2, b = \sqrt{2}\), then from (6) the \(\alpha\xi\)-pseudohyperbolical Smarandache curve \(\beta\) is given by (see Figure 2)
\[
\beta(s^* (s)) = \frac{1}{2} \left((\sqrt{2} + 1) s^2 + 2\sqrt{2}, (\sqrt{2} + 1) s^2 - 2, (2\sqrt{2} + 2) s\right).
\]  

From Theorem 7, its frame \(\{\beta, T_\beta, \xi_\beta\}\) is given by
\[
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} = \begin{bmatrix}
\sqrt{2} & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & \sqrt{2}
\end{bmatrix} \begin{bmatrix}
\alpha \\
T' \\
\xi
\end{bmatrix},
\]  

and the corresponding geodesic curvature \(k_\beta\) reads
\[
k_\beta = -1.
\]  

**Case 2.** If we take \(a = \sqrt{3}, b = -1\), then from (7) the \(\alpha T\)-pseudohyperbolical Smarandache curve \(\beta\) is given by (see Figure 3)
\[
\beta(s^* (s)) = \frac{\sqrt{2}}{2} \left(2\sqrt{3}s^2 - 2s + 2\sqrt{3}, \sqrt{3}s^2 - 2s, 2\sqrt{3}s - 2\right).
\]  

According to Theorem 8, its frame \(\{\beta, T_\beta, \xi_\beta\}\) is given by
\[
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} = \begin{bmatrix}
\sqrt{6} & -\sqrt{2} & \sqrt{3} \\
\frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} \\
\frac{3}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{6}} & \frac{3}{\sqrt{6}}
\end{bmatrix} \begin{bmatrix}
\alpha \\
T' \\
\xi
\end{bmatrix},
\]  

and the corresponding geodesic curvature \(k_\beta\) reads
\[
k_\beta = -1.
\]  

**Case 3.** If \(a = 3, b = \sqrt{3}\), and \(c = \sqrt{3}\), then from (7) the \(\alpha T\xi\)-pseudohyperbolical Smarandache curve \(\beta\) is given by (see Figure 4)
\[
\beta(s^* (s)) = \frac{1}{2} \left((\sqrt{3} + 1) s^2 + 2s + 2\sqrt{3}, (\sqrt{3} + 1) s^2 + 2s - 2, (2\sqrt{3} + 2)s + 2\right).
\]  

The orthonormal Sabban frame \(\{\alpha(s), T(s), \xi(s)\}\) along the curve \(\alpha\) is given by
\[
\alpha(s) = \left(\frac{s^2}{2} + 1, \frac{s^2}{2}, s\right),
\]
\[
T(s) = \alpha'(s) = (s, s, 1),
\]
\[
\xi(s) = \alpha(s) \times T(s) = \left(\frac{s^2}{2}, \frac{s^2}{2} - 1, s\right).
\]  

In particular, the geodesic curvature \(k_\alpha\) of the curve \(\alpha\) has the form
\[
k_\alpha(s) = -1.
\]  

**Case 1.** If we take \(a = 2, b = \sqrt{2}\), then from (6) the \(\alpha\xi\)-pseudohyperbolical Smarandache curve \(\beta\) is given by (see Figure 2)
\[
\beta(s^* (s)) = \frac{1}{2} \left((\sqrt{2} + 1) s^2 + 2\sqrt{2}, (\sqrt{2} + 1) s^2 - 2, (2\sqrt{2} + 2) s\right).
\]  

From Theorem 7, its frame \(\{\beta, T_\beta, \xi_\beta\}\) is given by
\[
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} = \begin{bmatrix}
\sqrt{2} & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & \sqrt{2}
\end{bmatrix} \begin{bmatrix}
\alpha \\
T' \\
\xi
\end{bmatrix},
\]  

and the corresponding geodesic curvature \(k_\beta\) reads
\[
k_\beta = -1.
\]  

**Case 2.** If we take \(a = \sqrt{3}, b = -1\), then from (7) the \(\alpha T\)-pseudohyperbolical Smarandache curve \(\beta\) is given by (see Figure 3)
\[
\beta(s^* (s)) = \frac{\sqrt{2}}{2} \left(2\sqrt{3}s^2 - 2s + 2\sqrt{3}, \sqrt{3}s^2 - 2s, 2\sqrt{3}s - 2\right).
\]  

According to Theorem 8, its frame \(\{\beta, T_\beta, \xi_\beta\}\) is given by
\[
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} = \begin{bmatrix}
\sqrt{6} & -\sqrt{2} & \sqrt{3} \\
\frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} \\
\frac{3}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{6}} & \frac{3}{\sqrt{6}}
\end{bmatrix} \begin{bmatrix}
\alpha \\
T' \\
\xi
\end{bmatrix},
\]  

and the corresponding geodesic curvature \(k_\beta\) reads
\[
k_\beta = -1.
\]  

**Case 3.** If \(a = 3, b = \sqrt{3}\), and \(c = \sqrt{3}\), then from (7) the \(\alpha T\xi\)-pseudohyperbolical Smarandache curve \(\beta\) is given by (see Figure 4)
\[
\beta(s^* (s)) = \frac{1}{2} \left((\sqrt{3} + 1) s^2 + 2s + 2\sqrt{3}, (\sqrt{3} + 1) s^2 + 2s - 2, (2\sqrt{3} + 2)s + 2\right).
\]
By using Theorem 9, it follows that the frame \( \{\beta, T_\beta, \xi_\beta\} \) is given by (see Figure 4)

\[
\begin{bmatrix}
\beta \\
T_\beta \\
\xi_\beta
\end{bmatrix} =
\begin{bmatrix}
\sqrt{3} & 1 & 1 \\
\frac{\sqrt{3} - 1}{2} & 1 & 1 - \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3} + 1}{2} & 1 & \frac{\sqrt{3} + 1}{2}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
T \\
\xi
\end{bmatrix},
\]  

(51)

and the corresponding geodesic curvature \( k_\beta \) reads

\[
k_\beta = -6 \left( 2 + \sqrt{3} \right) \left( 3 + \sqrt{3} \right)^{1/2}.
\]  

(52)

Also, Pseudohyperbolical Smarandache curves of \( \alpha \) and the curve \( \alpha \) on \( H_0^2 \) (1) with Figure 5 are shown.

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References

