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Article

# Neutrosophic Quadruple *BCK*/*BCI*-Algebras

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**Abstract:** The notion of a neutrosophic quadruple $BCK/BCI$-number is considered, and a neutrosophic quadruple $BCK/BCI$-algebra, which consists of neutrosophic quadruple $BCK/BCI$-numbers, is constructed. Several properties are investigated, and a (positive implicative) ideal in a neutrosophic quadruple $BCK$-algebra and a closed ideal in a neutrosophic quadruple $BCI$-algebra are studied. Given subsets $A$ and $B$ of a $BCK/BCI$-algebra, the set $NQ(A, B)$, which consists of neutrosophic quadruple $BCK/BCI$-numbers with a condition, is established. Conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple $BCK$-algebra are provided, and conditions for the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple $BCI$-algebra are given. An example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra is provided, and conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra are then discussed.

**Keywords:** neutrosophic quadruple $BCK/BCI$-number; neutrosophic quadruple $BCK/BCI$-algebra; neutrosophic quadruple subalgebra; (positive implicative) neutrosophic quadruple ideal

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## 1. Introduction

The notion of a neutrosophic set was developed by Smarandache [1–3] and is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets, and interval valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to a different field (see [4–8]). Neutrosophic algebraic structures in $BCK/BCI$-algebras are discussed in [9–16]. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [17,18].

In this paper, we will use neutrosophic quadruple numbers based on a set and construct neutrosophic quadruple $BCK/BCI$-algebras. We investigate several properties and consider ideals and positive implicative ideals in neutrosophic quadruple $BCK$-algebra, and closed ideals in neutrosophic quadruple $BCI$-algebra. Given subsets $A$ and $B$ of a neutrosophic quadruple $BCK/BCI$-algebra, we consider sets $NQ(A, B)$, which consist of neutrosophic quadruple $BCK/BCI$-numbers with a condition. We provide conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple $BCK$-algebra and for the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple $BCI$-algebra. We give an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra, and we then consider conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra.

## 2. Preliminaries

A $BCK/BCI$-algebra is an important class of logical algebras introduced by Iséki (see [19,20]).

By a $BCI$-algebra, we mean a set $X$ with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$;
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$;
(III) $(\forall x \in X)\ (x * x = 0)$;
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a $BCI$-algebra $X$ satisfies the identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then $X$ is called a *$BCK$-algebra*. Any $BCK/BCI$-algebra $X$ satisfies the following conditions:

$$(\forall x \in X)\ (x * 0 = x) \tag{1}$$
$$(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x) \tag{2}$$
$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y) \tag{3}$$
$$(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y) \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$. Any $BCI$-algebra $X$ satisfies the following conditions (see [21]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{5}$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{6}$$

A $BCK$-algebra $X$ is said to be *positive implicative* if the following assertion is valid.

$$(\forall x, y, z \in X)\ ((x * z) * (y * z) = (x * y) * z)\,. \tag{7}$$

A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. A subset $I$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ if it satisfies

$$0 \in I, \tag{8}$$
$$(\forall x \in X)\ (\forall y \in I)\ (x * y \in I\ \Rightarrow\ x \in I)\,. \tag{9}$$

A subset $I$ of a $BCI$-algebra $X$ is called a *closed ideal* (see [21]) of $X$ if it is an ideal of $X$ which satisfies

$$(\forall x \in X)(x \in I\ \Rightarrow\ 0 * x \in I). \tag{10}$$

A subset $I$ of a $BCK$-algebra $X$ is called a *positive implicative ideal* (see [22]) of $X$ if it satisfies (8) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\ y * z \in I\ \Rightarrow\ x * z \in I)\,. \tag{11}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [22]). Note also that a $BCK$-algebra $X$ is positive implicative if and only if every ideal of $X$ is positive implicative (see [22]).

We refer the reader to the books [21,22] for further information regarding $BCK/BCI$-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

## 3. Neutrosophic Quadruple *BCK/BCI*-Algebras

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

**Definition 1.** *Let $X$ be a set. A neutrosophic quadruple $X$-number is an ordered quadruple $(a, xT, yI, zF)$ where $a, x, y, z \in X$ and $T$, $I$, $F$ have their usual neutrosophic logic meanings.*

The set of all neutrosophic quadruple $X$-numbers is denoted by $NQ(X)$, that is,

$$NQ(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on $X$. If $X$ is a $BCK/BCI$-algebra, a neutrosophic quadruple $X$-number is called a *neutrosophic quadruple $BCK/BCI$-number* and we say that $NQ(X)$ is the *neutrosophic quadruple $BCK/BCI$-set*.

Let $X$ be a $BCK/BCI$-algebra. We define a binary operation $\odot$ on $NQ(X)$ by

$$(a, xT, yI, zF) \odot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all $(a, xT, yI, zF)$, $(b, uT, vI, wF) \in NQ(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple $BCK/BCI$-number $(a_1, a_2T, a_3I, a_4F)$ is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple $BCK/BCI$-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(X)$ as follows:

$$\begin{aligned} \tilde{x} \ll \tilde{y} &\Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4 \\ \tilde{x} = \tilde{y} &\Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4 \end{aligned}$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(X)$.

**Theorem 1.** *If $X$ is a $BCK/BCI$-algebra, then $(NQ(X); \odot, \tilde{0})$ is a $BCK/BCI$-algebra.*

**Proof.** Let $X$ be a $BCI$-algebra. For any $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$, we have

$$\begin{aligned} (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z}) &= (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\ &\qquad \odot (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \\ &= ((x_1 * y_1) * (x_1 * z_1), ((x_2 * y_2) * (x_2 * z_2))T, \\ &\qquad ((x_3 * y_3) * (x_3 * z_3))I, ((x_4 * y_4) * (x_4 * z_4))T) \\ &\ll (z_1 * y_1, (z_2 * y_2)T, (z_3 * y_3)I, (z_4 * y_4)F) \\ &= \tilde{z} \odot \tilde{y} \end{aligned}$$

$$\begin{aligned} \tilde{x} \odot (\tilde{x} \odot \tilde{y}) &= (x_1, x_2T, x_3I, x_4F) \odot (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\ &= (x_1 * (x_1 * y_1), (x_2 * (x_2 * y_2))T, (x_3 * (x_3 * y_3))I, (x_4 * (x_4 * y_4))F) \\ &\ll (y_1, y_2T, y_3I, y_4F) \\ &= \tilde{y} \end{aligned}$$

$$\begin{aligned} \tilde{x} \odot \tilde{x} &= (x_1, x_2T, x_3I, x_4F) \odot (x_1, x_2T, x_3I, x_4F) \\ &= (x_1 * x_1, (x_2 * x_2)T, (x_3 * x_3)I, (x_4 * x_4)F) \\ &= (0, 0T, 0I, 0F) = \tilde{0}. \end{aligned}$$

Assume that $\tilde{x} \odot \tilde{y} = \tilde{0}$ and $\tilde{y} \odot \tilde{x} = \tilde{0}$. Then

$$(x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) = (0, 0T, 0I, 0F)$$

and

$$(y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) = (0, 0T, 0I, 0F).$$

It follows that $x_1 * y_1 = 0 = y_1 * x_1$, $x_2 * y_2 = 0 = y_2 * x_2$, $x_3 * y_3 = 0 = y_3 * x_3$ and $x_4 * y_4 = 0 = y_4 * x_4$. Hence, $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, and $x_4 = y_4$, which implies that

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) = (y_1, y_2T, y_3I, y_4F) = \tilde{y}.$$

Therefore, we know that $(NQ(X); \odot, \tilde{0})$ is a $BCI$-algebra. We call it the *neutrosophic quadruple $BCI$-algebra*. Moreover, if $X$ is a $BCK$-algebra, then we have

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) = (0, 0T, 0I, 0F) = \tilde{0}.$$

Hence, $(NQ(X); \odot, \tilde{0})$ is a $BCK$-algebra. We call it the *neutrosophic quadruple BCK-algebra*. □

**Example 1.** *If $X = \{0, a\}$, then the neutrosophic quadruple set $NQ(X)$ is given as follows:*

$$NQ(X) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

*where*

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, aF)$, $\tilde{2} = (0, 0T, aI, 0F)$, $\tilde{3} = (0, 0T, aI, aF)$,
$\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, aF)$, $\tilde{6} = (0, aT, aI, 0F)$, $\tilde{7} = (0, aT, aI, aF)$,
$\tilde{8} = (a, 0T, 0I, 0F)$, $\tilde{9} = (a, 0T, 0I, aF)$, $\tilde{10} = (a, 0T, aI, 0F)$, $\tilde{11} = (a, 0T, aI, aF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, aF)$, $\tilde{14} = (a, aT, aI, 0F)$, *and* $\tilde{15} = (a, aT, aI, aF)$.

*Consider a BCK-algebra $X = \{0, a\}$ with the binary operation $*$, which is given in Table 1.*

**Table 1.** Cayley table for the binary operation "$*$".

| $*$ | **0** | $\boldsymbol{a}$ |
|---|---|---|
| 0 | 0 | 0 |
| $a$ | $a$ | 0 |

*Then $(NQ(X), \odot, \tilde{0})$ is a BCK-algebra in which the operation $\odot$ is given by Table 2.*

**Table 2.** Cayley table for the binary operation "$\odot$".

| $\odot$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{2}$ | $\tilde{3}$ | $\tilde{4}$ | $\tilde{5}$ | $\tilde{6}$ | $\tilde{7}$ | $\tilde{8}$ | $\tilde{9}$ | $\tilde{10}$ | $\tilde{11}$ | $\tilde{12}$ | $\tilde{13}$ | $\tilde{14}$ | $\tilde{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{1}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{2}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{3}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{5}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{6}$ | $\tilde{6}$ | $\tilde{6}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{6}$ | $\tilde{6}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{7}$ | $\tilde{7}$ | $\tilde{6}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{7}$ | $\tilde{6}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{9}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{9}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{10}$ | $\tilde{10}$ | $\tilde{10}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{10}$ | $\tilde{10}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ |

**Table 2.** *Cont.*

| $\odot$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{2}$ | $\tilde{3}$ | $\tilde{4}$ | $\tilde{5}$ | $\tilde{6}$ | $\tilde{7}$ | $\tilde{8}$ | $\tilde{9}$ | $\tilde{10}$ | $\tilde{11}$ | $\tilde{12}$ | $\tilde{13}$ | $\tilde{14}$ | $\tilde{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tilde{11}$ | $\tilde{11}$ | $\tilde{10}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{11}$ | $\tilde{10}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{12}$ | $\tilde{12}$ | $\tilde{12}$ | $\tilde{12}$ | $\tilde{12}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{13}$ | $\tilde{13}$ | $\tilde{12}$ | $\tilde{13}$ | $\tilde{12}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{1}$ | $\tilde{0}$ | $\tilde{1}$ | $\tilde{0}$ |
| $\tilde{14}$ | $\tilde{14}$ | $\tilde{14}$ | $\tilde{12}$ | $\tilde{12}$ | $\tilde{10}$ | $\tilde{10}$ | $\tilde{8}$ | $\tilde{8}$ | $\tilde{6}$ | $\tilde{6}$ | $\tilde{4}$ | $\tilde{4}$ | $\tilde{2}$ | $\tilde{2}$ | $\tilde{0}$ | $\tilde{0}$ |
| $\tilde{15}$ | $\tilde{15}$ | $\tilde{14}$ | $\tilde{13}$ | $\tilde{12}$ | $\tilde{11}$ | $\tilde{10}$ | $\tilde{9}$ | $\tilde{8}$ | $\tilde{7}$ | $\tilde{6}$ | $\tilde{5}$ | $\tilde{4}$ | $\tilde{3}$ | $\tilde{2}$ | $\tilde{1}$ | $\tilde{0}$ |

**Theorem 2.** *The neutrosophic quadruple set $NQ(X)$ based on a positive implicative BCK-algebra $X$ is a positive implicative BCK-algebra.*

**Proof.** Let $X$ be a positive implicative $BCK$-algebra. Then $X$ is a $BCK$-algebra, so $(NQ(X); \odot, \tilde{0})$ is a $BCK$-algebra by Theorem 1. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Then

$$(x_i * z_i) * (y_i * z_i) = (x_i * y_i) * z_i$$

for all $i = 1, 2, 3, 4$ since $x_i, y_i, z_i \in X$ and $X$ is a positive implicative $BCK$-algebra. Hence, $(\tilde{x} \odot \tilde{z}) \odot (\tilde{y} * \tilde{z}) = (\tilde{x} \odot \tilde{y}) \odot \tilde{z}$; therefore, $NQ(X)$ based on a positive implicative $BCK$-algebra $X$ is a positive implicative $BCK$-algebra. □

**Proposition 1.** *The neutrosophic quadruple set $NQ(X)$ based on a positive implicative BCK-algebra $X$ satisfies the following assertions.*

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) \ (\tilde{x} \odot \tilde{y} \ll \tilde{z} \Rightarrow \tilde{x} \odot \tilde{z} \ll \tilde{y} \odot \tilde{z}) \quad (12)$$

$$(\forall \tilde{x}, \tilde{y} \in NQ(X)) \ (\tilde{x} \odot \tilde{y} \ll \tilde{y} \Rightarrow \tilde{x} \ll \tilde{y}). \quad (13)$$

**Proof.** Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. If $\tilde{x} \odot \tilde{y} \ll \tilde{z}$, then

$$\tilde{0} = (\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}),$$

so $\tilde{x} \odot \tilde{z} \ll \tilde{y} \odot \tilde{z}$. Assume that $\tilde{x} \odot \tilde{y} \ll \tilde{y}$. Using Equation (12) implies that

$$\tilde{x} \odot \tilde{y} \ll \tilde{y} \odot \tilde{y} = \tilde{0},$$

so $\tilde{x} \odot \tilde{y} = \tilde{0}$, i.e., $\tilde{x} \ll \tilde{y}$. □

Let $X$ be a $BCK/BCI$-algebra. Given $a, b \in X$ and subsets $A$ and $B$ of $X$, consider the sets

$$NQ(a, B) := \{(a, aT, yI, zF) \in NQ(X) \mid y, z \in B\}$$

$$NQ(A, b) := \{(a, xT, bI, bF) \in NQ(X) \mid a, x \in A\}$$

$$NQ(A, B) := \{(a, xT, yI, zF) \in NQ(X) \mid a, x \in A; y, z \in B\}$$

$$NQ(A^*, B) := \bigcup_{a \in A} NQ(a, B)$$

$$NQ(A, B^*) := \bigcup_{b \in B} NQ(A, b)$$

and

$$NQ(A \cup B) := NQ(A, 0) \cup NQ(0, B).$$

The set $NQ(A, A)$ is denoted by $NQ(A)$.

**Proposition 2.** *Let $X$ be a $BCK/BCI$-algebra. Given $a, b \in X$ and subsets $A$ and $B$ of $X$, we have*

(1) *$NQ(A^*, B)$ and $NQ(A, B^*)$ are subsets of $NQ(A, B)$.*
(1) *If $0 \in A \cap B$ then $NQ(A \cup B)$ is a subset of $NQ(A, B)$.*

**Proof.** Straightforward. □

Let $X$ be a $BCK/BCI$-algebra. Given $a, b \in X$ and subalgebras $A$ and $B$ of $X$, $NQ(a, B)$ and $NQ(A, b)$ may not be subalgebras of $NQ(X)$ since

$$(a, aT, x_3I, x_4F) \odot (a, aT, u_3I, v_4F) = (0, 0T, (x_3 * u_3)I, (x_4 * v_4)F) \notin NQ(a, B)$$

and

$$(x_1, x_2T, bI, bF) \odot (u_1, u_2T, bI, bF) = (x_1 * u_1, (x_2 * u_2)T, 0I, 0F) \notin NQ(A, b)$$

for $(a,\ aT,\ x_3I,\ x_4F) \in NQ(a, B)$, $(a,\ aT,\ u_3I,\ v_4F) \in NQ(a, B)$, $(x_1,\ x_2T,\ bI,\ bF) \in NQ(A, b)$, and $(u_1, u_2T, bI, bF) \in NQ(A, b)$.

**Theorem 3.** *If $A$ and $B$ are subalgebras of a $BCK/BCI$-algebra $X$, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$, which is called a neutrosophic quadruple subalgebra.*

**Proof.** Assume that $A$ and $B$ are subalgebras of a $BCK/BCI$-algebra X. Let $\tilde{x} = (x_1,\ x_2T,\ x_3I,\ x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(A, B)$. Then $x_1$, $x_2$, $y_1$, $y_2 \in A$ and $x_3$, $x_4$, $y_3$, $y_4 \in B$, which implies that $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $NQ(A, B)$ is a subalgebra of $NQ(X)$. □

**Theorem 4.** *If $A$ and $B$ are ideals of a $BCK/BCI$-algebra $X$, then the set $NQ(A, B)$ is an ideal of $NQ(X)$, which is called a neutrosophic quadruple ideal.*

**Proof.** Assume that $A$ and $B$ are ideals of a $BCK/BCI$-algebra $X$. Obviously, $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1,\ x_2T,\ x_3I,\ x_4F)$ and $\tilde{y} = (y_1,\ y_2T,\ y_3I,\ y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$ and $x_4 * y_4 \in B$. Since $\tilde{y} \in NQ(A, B)$, we have $y_1, y_2 \in A$ and $y_3, y_4 \in B$. Since $A$ and $B$ are ideals of $X$, it follows that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence, $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$, so $NQ(A, B)$ is an ideal of $NQ(X)$. □

Since every ideal is a subalgebra in a $BCK$-algebra, we have the following corollary.

**Corollary 1.** *If $A$ and $B$ are ideals of a $BCK$-algebra $X$, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$.*

The following example shows that Corollary 1 is not true in a $BCI$-algebra.

**Example 2.** *Consider a $BCI$-algebra $(\mathbb{Z}, -, 0)$. If we take $A = \mathbb{N}$ and $B = \mathbb{Z}$, then $NQ(A, B)$ is an ideal of $NQ(\mathbb{Z})$. However, it is not a subalgebra of $NQ(\mathbb{Z})$ since*

$$(2, 3T, -5I, 6F) \odot (3, 5T, 6I, -7F) = (-1, -2T, -11I, 13F) \notin NQ(A, B)$$

*for $(2, 3T, -5I, 6F), (3, 5T, 6I, -7F) \in NQ(A, B)$.*

**Theorem 5.** *If $A$ and $B$ are closed ideals of a $BCI$-algebra $X$, then the set $NQ(A, B)$ is a closed ideal of $NQ(X)$.*

**Proof.** If $A$ and $B$ are closed ideals of a $BCI$-algebra $X$, then the set $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Then

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \in NQ(A, B)$$

since $0 * x_1, 0 * x_2 \in A$ and $0 * x_3, 0 * x_4 \in B$. Therefore, $NQ(A, B)$ is a closed ideal of $NQ(X)$. □

Since every closed ideal of a $BCI$-algebra $X$ is a subalgebra of $X$, we have the following corollary.

**Corollary 2.** *If $A$ and $B$ are closed ideals of a $BCI$-algebra $X$, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$.*

In the following example, we know that there exist ideals $A$ and $B$ in a $BCI$-algebra $X$ such that $NQ(A, B)$ is not a closed ideal of $NQ(X)$.

**Example 3.** *Consider $BCI$-algebras $(Y, *, 0)$ and $(\mathbb{Z}, -, 0)$. Then $X = Y \times \mathbb{Z}$ is a $BCI$-algebra (see [21]). Let $A = Y \times \mathbb{N}$ and $B = \{0\} \times \mathbb{N}$. Then $A$ and $B$ are ideals of $X$, so $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $((0, 0), (0, 1)T, (0, 2)I, (0, 3)F) \in NQ(A, B)$. Then*

$$\begin{aligned} &((0, 0), (0, 0)T, (0, 0)I, (0, 0)F) \odot ((0, 0), (0, 1)T, (0, 2)I, (0, 3)F) \\ &= ((0, 0), (0, -1)T, (0, -2)I, (0, -3)F) \notin NQ(A, B). \end{aligned}$$

*Hence, $NQ(A, B)$ is not a closed ideal of $NQ(X)$.*

We provide conditions wherethe set $NQ(A, B)$ is a closed ideal of $NQ(X)$.

**Theorem 6.** *Let $A$ and $B$ be ideals of a $BCI$-algebra $X$ and let*

$$\Gamma := \{\tilde{a} \in NQ(X) \mid (\forall \tilde{x} \in NQ(X))(\tilde{x} \ll \tilde{a} \ \Rightarrow \ \tilde{x} = \tilde{a})\}.$$

*Assume that, if $\Gamma \subseteq NQ(A, B)$, then $|\Gamma| < \infty$. Then $NQ(A, B)$ is a closed ideal of $NQ(X)$.*

**Proof.** If $A$ and $B$ are ideals of $X$, then $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{a} = (a_1, a_2T, a_3I, a_4F) \in NQ(A, B)$. For any $n \in \mathbb{N}$, denote $n(\tilde{a}) := \tilde{0} \odot (\tilde{0} \odot \tilde{a})^n$. Then $n(\tilde{a}) \in \Gamma$ and

$$\begin{aligned} n(\tilde{a}) &= (0 * (0 * a_1)^n, (0 * (0 * a_2)^n)T, (0 * (0 * a_3)^n)I, (0 * (0 * a_4)^n)F) \\ &= (0 * (0 * a_1^n), (0 * (0 * a_2^n))T, (0 * (0 * a_3^n))I, (0 * (0 * a_4^n))F) \\ &= \tilde{0} \odot (\tilde{0} \odot \tilde{a}^n). \end{aligned}$$

Hence,

$$\begin{aligned} n(\tilde{a}) \odot \tilde{a}^n &= (\tilde{0} \odot (\tilde{0} \odot \tilde{a}^n)) \odot \tilde{a}^n \\ &= (\tilde{0} \odot \tilde{a}^n) \odot (\tilde{0} \odot \tilde{a}^n) \\ &= \tilde{0} \in NQ(A, B), \end{aligned}$$

so $n(\tilde{a}) \in NQ(A,B)$, since $\tilde{a} \in NQ(A,B)$, and $NQ(A,B)$ is an ideal of $NQ(X)$. Since $|\Gamma| < \infty$, it follows that $k \in \mathbb{N}$ such that $n(\tilde{a}) = (n+k)(\tilde{a})$, that is, $n(\tilde{a}) = n(\tilde{a}) \odot (\tilde{0} \odot \tilde{a})^k$, and thus

$$\begin{aligned} k(\tilde{a}) &= \tilde{0} \odot (\tilde{0} \odot \tilde{a})^k \\ &= (n(\tilde{a}) \odot (\tilde{0} \odot \tilde{a})^k) \odot n(\tilde{a}) \\ &= n(\tilde{a}) \odot n(\tilde{a}) = \tilde{0}, \end{aligned}$$

i.e., $(k-1)(\tilde{a}) \odot (\tilde{0} \odot \tilde{a}) = \tilde{0}$. Since $\tilde{0} \odot \tilde{a} \in \Gamma$, it follows that $\tilde{0} \odot \tilde{a} = (k-1)(\tilde{a}) \in NQ(A,B)$. Therefore, $NQ(A,B)$ is a closed ideal of $NQ(X)$. □

**Theorem 7.** *Given two elements $a$ and $b$ in a $BCI$-algebra $X$, let*

$$A_a := \{x \in X \mid a * x = a\} \text{ and } B_b := \{x \in X \mid b * x = b\}. \tag{14}$$

*Then $NQ(A_a, B_b)$ is a closed ideal of $NQ(X)$.*

**Proof.** Since $a * 0 = a$ and $b * 0 = b$, we have $0 \in A_a \cap B_b$. Thus, $\tilde{0} \in NQ(A_a, B_b)$. If $x \in A_a$ and $y \in B_b$, then

$$0 * x = (a * x) * a = a * a = 0 \text{ and } 0 * y = (b * y) * b = b * b = 0. \tag{15}$$

Let $x, y, c, d \in X$ be such that $x, y * x \in A_a$ and $c, d * c \in B_b$. Then

$$(a * y) * a = 0 * y = (0 * y) * 0 = (0 * y) * (0 * x) = 0 * (y * x) = 0$$

and

$$(b * d) * b = 0 * d = (0 * d) * 0 = (0 * d) * (0 * c) = 0 * (d * c) = 0,$$

that is, $a * y \leq a$ and $b * d \leq b$. On the other hand,

$$a = a * (y * x) = (a * x) * (y * x) \leq a * y$$

and

$$b = b * (d * c) = (b * c) * (d * c) \leq b * d.$$

Thus, $a * y = a$ and $b * d = b$, i.e., $y \in A_a$ and $d \in B_b$. Hence, $A_a$ and $B_b$ are ideals of $X$, and $NQ(A_a, B_b)$ is therefore an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A_a, B_b)$. Then $x_1, x_2 \in A_a$, and $x_3, x_4 \in B_b$. It follows from Equation (15) that $0 * x_1 = 0 \in A_a$, $0 * x_2 = 0 \in A_a$, $0 * x_3 = 0 \in B_b$, and $0 * x_4 = 0 \in B_b$. Hence,

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \in NQ(A_a, B_b).$$

Therefore, $NQ(A_a, B_b)$ is a closed ideal of $NQ(X)$. □

**Proposition 3.** *Let $A$ and $B$ be ideals of a BCK-algebra $X$. Then*

$$NQ(A) \cap NQ(B) = \{\tilde{0}\} \;\Leftrightarrow\; (\forall \tilde{x} \in NQ(A))(\forall \tilde{y} \in NQ(B))(\tilde{x} \odot \tilde{y} = \tilde{x}). \tag{16}$$

**Proof.** Note that $NQ(A)$ and $NQ(B)$ are ideals of $NQ(X)$. Assume that $NQ(A) \cap NQ(B) = \{\tilde{0}\}$. Let

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A) \text{ and } \tilde{y} = (y_1, y_2T, y_3I, y_4F) \in NQ(B).$$

Since $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \ll \tilde{x}$ and $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \ll \tilde{y}$, it follows that $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \in NQ(A) \cap NQ(B) = \{\tilde{0}\}$. Obviously, $(\tilde{x} \odot \tilde{y}) \odot \tilde{x} \in \{\tilde{0}\}$. Hence, $\tilde{x} \odot \tilde{y} = \tilde{x}$.

Conversely, suppose that $\tilde{x} \odot \tilde{y} = \tilde{x}$ for all $\tilde{x} \in NQ(A)$ and $\tilde{y} \in NQ(B)$. If $\tilde{z} \in NQ(A) \cap NQ(B)$, then $\tilde{z} \in NQ(A)$ and $\tilde{z} \in NQ(B)$, which is implied from the hypothesis that $\tilde{z} = \tilde{z} \odot \tilde{z} = \tilde{0}$. Hence $NQ(A) \cap NQ(B) = \{\tilde{0}\}$. □

**Theorem 8.** *Let $A$ and $B$ be subsets of a BCK-algebra $X$ such that*

$$(\forall a, b \in A \cap B)(K(a, b) \subseteq A \cap B) \tag{17}$$

*where $K(a, b) := \{x \in X \mid x * a \leq b\}$. Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.*

**Proof.** If $x \in A \cap B$, then $0 \in K(x, x)$ since $0 * x \leq x$. Hence, $0 \in A \cap B$ by Equation (17), so it is clear that $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Using (II), we have $x_1 \in K(x_1 * y_1, y_1) \subseteq A$, $x_2 \in K(x_2 * y_2, y_2) \subseteq A$, $x_3 \in K(x_3 * y_3, y_3) \subseteq B$, and $x_4 \in K(x_4 * y_4, y_4) \subseteq B$. This implies that $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Therefore, $NQ(A, B)$ is an ideal of $NQ(X)$. □

**Corollary 3.** *Let $A$ and $B$ be subsets of a BCK-algebra $X$ such that*

$$(\forall a, x, y \in X)(x, y \in A \cap B,\ (a * x) * y = 0 \ \Rightarrow \ a \in A \cap B). \tag{18}$$

*Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.*

**Theorem 9.** *Let $A$ and $B$ be nonempty subsets of a BCK-algebra $X$ such that*

$$(\forall a, x, y \in X)(x, y \in A \text{ (or } B),\ a * x \leq y \ \Rightarrow \ a \in A \text{ (or } B)). \tag{19}$$

*Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.*

**Proof.** Assume that the condition expressed by Equation (19) is valid for nonempty subsets $A$ and $B$ of $X$. Since $0 * x \leq x$ for any $x \in A$ (or $B$), we have $0 \in A$ (or $B$) by Equation (19). Hence, it is clear that $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Note that $x_i * (x_i * y_i) \leq y_i$ for $i = 1, 2, 3, 4$. It follows from Equation (19) that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence,

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B);$$

therefore, $NQ(A, B)$ is an ideal of $NQ(X)$. □

**Theorem 10.** *If $A$ and $B$ are positive implicative ideals of a BCK-algebra $X$, then the set $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$, which is called a positive implicative neutrosophic quadruple ideal.*

**Proof.** Assume that $A$ and $B$ are positive implicative ideals of a $BCK$-algebra $X$. Obviously, $\tilde{0} \in NQ(A,B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A,B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A,B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T,\\ ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A,B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A,B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. Since $A$ and $B$ are positive implicative ideals of $X$, it follows that $x_1 * z_1, x_2 * z_2 \in A$ and $x_3 * z_3, x_4 * z_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A,B),$$

so $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$. □

**Theorem 11.** *Let $A$ and $B$ be ideals of a $BCK$-algebra $X$ such that*

$$(\forall x,y,z \in X)((x * y) * z \in A \text{ (or } B) \;\Rightarrow\; (x * z) * (y * z) \in A \text{ (or } B)). \tag{20}$$

*Then $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$.*

**Proof.** Since $A$ and $B$ are ideals of $X$, it follows from Theorem 4 that $NQ(A,B)$ is an ideal of $NQ(X)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A,B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A,B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T,\\ ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A,B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A,B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. It follows from Equation (20) that $(x_1 * z_1) * (y_1 * z_1) \in A$, $(x_2 * z_2) * (y_2 * z_2) \in A$, $(x_3 * z_3) * (y_3 * z_3) \in B$, and $(x_4 * z_4) * (y_4 * z_4) \in B$. Since $A$ and $B$ are ideals of $X$, we get $x_1 * z_1 \in A$, $x_2 * z_2 \in A$, $x_3 * z_3 \in B$, and $x_4 * z_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A,B).$$

Therefore, $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$. □

**Corollary 4.** *Let $A$ and $B$ be ideals of a $BCK$-algebra $X$ such that*

$$(\forall x,y \in X)((x * y) * y \in A \text{ (or } B) \;\Rightarrow\; x * y \in A \text{ (or } B)). \tag{21}$$

*Then $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$.*

**Proof.** If the condition expressed in Equation (21) is valid, then the condition expressed in Equation (20) is true. Hence, $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$ by Theorem 11. □

**Theorem 12.** *Let $A$ and $B$ be subsets of a BCK-algebra $X$ such that $0 \in A \cap B$ and*

$$((x * y) * y) * z \in A \text{ (or } B), \; z \in A \text{ (or } B) \;\Rightarrow\; x * y \in A \text{ (or } B) \tag{22}$$

*for all $x, y, z \in X$. Then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.*

**Proof.** Since $0 \in A \cap B$, it is clear that $\tilde{0} \in NQ(A, B)$. We first show that

$$(\forall x, y \in X)(x * y \in A \text{ (or } B), \; y \in A \text{ (or } B) \Rightarrow x \in A \text{ (or } B)). \tag{23}$$

Let $x, y \in X$ be such that $x * y \in A$ (or $B$) and $y \in A$ (or $B$). Then

$$((x * 0) * 0) * y = x * y \in A \text{ (or } B)$$

by Equation (1), which, based on Equations (1) and (22), implies that $x = x * 0 \in A$ (or $B$). Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (&(x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, \\ &((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A, B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A, B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. Note that

$$(((x_i * z_i) * z_i) * (y_i * z_i)) * ((x_i * y_i) * z_i) = 0 \in A \text{ (or } B)$$

for $i = 1, 2, 3, 4$. Since $(x_i * y_i) * z_i \in A$ for $i = 1, 2$ and $(x_j * y_j) * z_j \in B$ for $j = 3, 4$, it follows from Equation (23) that $((x_i * z_i) * z_i) * (y_i * z_i) \in A$ for $i = 1, 2$, and $((x_j * z_j) * z_j) * (y_j * z_j) \in B$ for $j = 3, 4$. Moreover, since $y_i * z_i \in A$ for $i = 1, 2$, and $y_j * z_j \in B$ for $j = 3, 4$, we have $x_1 * z_1 \in A$, $x_2 * z_2 \in A$, $x_3 * z_3 \in B$, and $x_4 * z_4 \in B$ by Equation (22). Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A, B).$$

Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

**Theorem 13.** *Let $A$ and $B$ be subsets of a BCK-algebra $X$ such that $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. Then the set*

$$\Omega_{\tilde{a}} := \{\tilde{x} \in NQ(X) \mid \tilde{x} \odot \tilde{a} \in NQ(A, B)\} \tag{24}$$

*is an ideal of $NQ(X)$ for any $\tilde{a} \in NQ(X)$.*

**Proof.** Obviously, $\tilde{0} \in \Omega_{\tilde{a}}$. Let $\tilde{x}, \tilde{y} \in NQ(X)$ be such that $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{a}}$ and $\tilde{y} \in \Omega_{\tilde{a}}$. Then $(\tilde{x} \odot \tilde{y}) \odot \tilde{a} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{a} \in NQ(A, B)$. Since $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$, it follows from Equation (11) that $\tilde{x} \odot \tilde{a} \in NQ(A, B)$ and therefore that $\tilde{x} \in \Omega_{\tilde{a}}$. Hence, $\Omega_{\tilde{a}}$ is an ideal of $NQ(X)$. □

Combining Theorems 12 and 13, we have the following corollary.

**Corollary 5.** *If $A$ and $B$ are subsets of a BCK-algebra $X$ satisfying $0 \in A \cap B$ and the condition expressed in Equation (22), then the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$.*

**Theorem 14.** *For any subsets $A$ and $B$ of a BCK-algebra $X$, if the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$, then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.*

**Proof.** Since $\tilde{0} \in \Omega_{\tilde{a}}$, we have $\tilde{0} = \tilde{0} \odot \tilde{a} \in NQ(A, B)$. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$ be such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{z}}$ and $\tilde{y} \in \Omega_{\tilde{z}}$. Since $\Omega_{\tilde{z}}$ is an ideal of $NQ(X)$, it follows that $\tilde{x} \in \Omega_{\tilde{z}}$. Hence, $\tilde{x} \odot \tilde{z} \in NQ(A, B)$. Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

**Theorem 15.** *For any ideals $A$ and $B$ of a BCK-algebra $X$ and for any $\tilde{a} \in NQ(X)$, if the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$, then $NQ(X)$ is a positive implicative BCK-algebra.*

**Proof.** Let $\Omega$ be any ideal of $NQ(X)$. For any $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$, assume that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in \Omega$ and $\tilde{y} \odot \tilde{z} \in \Omega$. Then $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{z}}$ and $\tilde{y} \in \Omega_{\tilde{z}}$. Since $\Omega_{\tilde{z}}$ is an ideal of $NQ(X)$, it follows that $\tilde{x} \in \Omega_{\tilde{z}}$. Hence, $\tilde{x} \odot \tilde{z} \in \Omega$, which shows that $\Omega$ is a positive implicative ideal of $NQ(X)$. Therefore, $NQ(X)$ is a positive implicative $BCK$-algebra. □

In general, the set $\{\tilde{0}\}$ is an ideal of any neutrosophic quadruple $BCK$-algebra $NQ(X)$, but it is not a positive implicative ideal of $NQ(X)$ as seen in the following example.

**Example 4.** *Consider a BCK-algebra $X = \{0, 1, 2\}$ with the binary operation $*$, which is given in Table 3.*

**Table 3.** Cayley table for the binary operation "$*$".

| $*$ | **0** | **1** | **2** |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 2 | 2 | 1 | 0 |

*Then the neutrosophic quadruple BCK-algebra $NQ(X)$ has 81 elements. If we take $\tilde{a} = (2, 2T, 2I, 2F)$ and $\tilde{b} = (1, 1T, 1I, 1F)$ in $NQ(X)$, then*

$$\begin{aligned}(\tilde{a} \odot \tilde{b}) \odot \tilde{b} &= ((2*1)*1, ((2*1)*1)T, ((2*1)*1)I, ((2*1)*1)F) \\ &= (1*1, (1*1)T, (1*1)I, (1*1)F) = (0, 0T, 0I, 0F) = \tilde{0},\end{aligned}$$

*and $\tilde{b} \odot \tilde{b} = \tilde{0}$. However,*

$$\tilde{a} \odot \tilde{b} = (2*1, (2*1)T, (2*1)I, (2*1)F) = (1, 1T, 1I, 1F) \neq \tilde{0}.$$

*Hence, $\{\tilde{0}\}$ is not a positive implicative ideal of $NQ(X)$.*

We now provide conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in the neutrosophic quadruple $BCK$-algebra.

**Theorem 16.** *Let $NQ(X)$ be a neutrosophic quadruple BCK-algebra. If the set*

$$\Omega(\tilde{a}) := \{\tilde{x} \in NQ(X) \mid \tilde{x} \ll \tilde{a}\} \tag{25}$$

*is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$, then $\{\tilde{0}\}$ is a positive implicative ideal of $NQ(X)$.*

**Proof.** We first show that

$$(\forall \tilde{x}, \tilde{y} \in NQ(X))((\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{0} \Rightarrow \tilde{x} \odot \tilde{y} = \tilde{0}). \tag{26}$$

Assume that $(\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{0}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Then $\tilde{x} \odot \tilde{y} \ll \tilde{y}$, so $\tilde{x} \odot \tilde{y} \in \Omega(\tilde{y})$. Since $\tilde{y} \in \Omega(\tilde{y})$ and $\Omega(\tilde{y})$ is an ideal of $NQ(X)$, we have $\tilde{x} \in \Omega(\tilde{y})$. Thus, $\tilde{x} \ll \tilde{y}$, that is, $\tilde{x} \odot \tilde{y} = \tilde{0}$. Let $\tilde{u} := (\tilde{x} \odot \tilde{y}) \odot \tilde{y}$. Then

$$((\tilde{x} \odot \tilde{u}) \odot \tilde{y}) \odot \tilde{y} = ((\tilde{x} \odot \tilde{y}) \odot \tilde{y}) \odot \tilde{u} = \tilde{0},$$

which implies, based on Equations (3) and (26), that

$$(\tilde{x} \odot \tilde{y}) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{y}) = (\tilde{x} \odot \tilde{y}) \odot \tilde{u} = (\tilde{x} \odot \tilde{u}) \odot \tilde{y} = \tilde{0},$$

that is, $\tilde{x} \odot \tilde{y} \ll (\tilde{x} \odot \tilde{y}) \odot \tilde{y}$. Since $(\tilde{x} \odot \tilde{y}) \odot \tilde{y} \ll \tilde{x} \odot \tilde{y}$, it follows that

$$(\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{x} \odot \tilde{y}. \tag{27}$$

If we put $\tilde{y} = \tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))$ in Equation (27), then

$$\begin{aligned}
\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) &= (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\
&= (\tilde{y} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}) \\
&= ((\tilde{y} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x}) \\
&\ll (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \\
&= ((\tilde{x} \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))))) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \\
&= ((\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot (\tilde{y} \odot \tilde{x})) = \tilde{0},
\end{aligned}$$

so $((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) = \tilde{0}$, that is,

$$((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \ll \tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))).$$

Hence,

$$\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) = ((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})). \tag{28}$$

If we use $\tilde{y} \odot \tilde{x}$ instead of $\tilde{x}$ in Equation (28), then

$$\begin{aligned}
\tilde{y} \odot \tilde{x} &= (\tilde{y} \odot \tilde{x}) \odot \tilde{0} \\
&= (\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \\
&= ((\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\
&= (\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})),
\end{aligned}$$

which, by taking $\tilde{x} = \tilde{y} \odot \tilde{x}$, implies that

$$\begin{aligned}\tilde{y} \odot (\tilde{y} \odot \tilde{x}) &= (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\ &= (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x}).\end{aligned}$$

It follows that

$$\begin{aligned}(\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) &= ((\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\ &\ll (\tilde{x} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\ &= (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}),\end{aligned}$$

so,

$$\begin{aligned}\tilde{y} \odot \tilde{x} &= (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot \tilde{0} \\ &= (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y}) \\ &\ll ((\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\ &= (\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\ &\ll (\tilde{y} \odot \tilde{x}) \odot \tilde{x}.\end{aligned}$$

Since $(\tilde{y} \odot \tilde{x}) \odot \tilde{x} \ll \tilde{y} \odot \tilde{x}$, it follows that

$$(\tilde{y} \odot \tilde{x}) \odot \tilde{x} = \tilde{y} \odot \tilde{x}. \tag{29}$$

Based on Equation (29), it follows that

$$\begin{aligned}&((\tilde{x} \odot \tilde{z}) * (\tilde{y} \odot \tilde{z})) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &= (((\tilde{x} \odot \tilde{z}) \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &\ll ((\tilde{x} \odot \tilde{z}) \odot \tilde{y}) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &= \tilde{0},\end{aligned}$$

that is, $(\tilde{x} \odot \tilde{z}) * (\tilde{y} \odot \tilde{z}) \ll (\tilde{x} \odot \tilde{y}) \odot \tilde{z}$. Note that

$$\begin{aligned}&((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \odot ((x \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})) \\ &= ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \odot ((x \odot (\tilde{y} \odot \tilde{z})) \odot \tilde{z}) \\ &\ll (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot (\tilde{y} \odot \tilde{z})) \\ &\ll (\tilde{y} \odot \tilde{z}) \odot \tilde{y} = \tilde{0},\end{aligned}$$

which shows that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \ll (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})$. Hence, $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})$. Therefore, $NQ(X)$ is a positive implicative, so $\{\tilde{0}\}$ is a positive implicative ideal of $NQ(X)$. □

## 4. Conclusions

We have considered a neutrosophic quadruple $BCK/BCI$-number on a set and established neutrosophic quadruple $BCK/BCI$-algebras, which consist of neutrosophic quadruple $BCK/BCI$-numbers. We have investigated several properties and considered ideal theory in a neutrosophic quadruple $BCK$-algebra and a closed ideal in a neutrosophic quadruple $BCI$-algebra. Using subsets $A$ and $B$ of a neutrosophic quadruple $BCK/BCI$-algebra, we have considered sets $NQ(A, B)$, which consist of neutrosophic quadruple $BCK/BCI$-numbers with a condition. We have provided conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple $BCK$-algebra, and the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple $BCI$-algebra. We have provided an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra, and we have considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra.

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