Compiling Adiabatic Quantum Programs

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***Abstract*—We develop a non-cooperative game-theoretic model for the problem of graph minor-embedding to show that optimal compiling of adiabatic quantum programs in the sense of Nash equilibrium is possible.**

***Index Terms*—Adiabatic quantum computing, non-cooperative game theory, minor-embedding, quantum compiling, network creation game**

I. Introduction

The adiabatic quantum computation (AQC) paradigm [1] is inspired by the physical processes known as quantum annealing where one starts with a quantum system in its lowest energy state and evolves the system "slowly enough" so that the new state is also that of lowest energy. More specifically, one initializes a quantum system so that its Hamiltonian H_I is in the lowest energy state, and then interpolates to a final or problem quantum system with Hamiltonian H_F via the *AQC program*

$$H(t) = A(t)H_I + B(t)H_F \tag{1}$$

with $A(0) = 1, B(0) = 0$ and $A(T) = 0, B(T) = 1$, where T is the run-time of the program $H(t)$. The value of T determines if the solution to a problem obtained using an AQC program is worthwhile in comparison to run-times of programs on standard computers. The value of T is determined inversely by the minimum spectral gap, which is the distance between the lowest energy state and the next excited state of the Hamiltonian $H(t)$. But this gap in turn may depend on how the program $H(t)$ is compiled for processing by an adiabatic quantum processor [2]. In the following sections, we consider the compilation of an AQC program as a type of network creation game [3] and show that an optimal compilation of AQC programs, in the sense of Nash equilibrium, is possible via fixed-point stability.

II. Adiabatic quantum programming

Compiling an AQC program $H(t)$ involves graph-theoretic considerations that are dependent on the hardware structure of the quantum processor. This is because the quantum hardware is effectively a graph, hereafter referred to as the *hardware graph* Γ_H, in which the vertices represent qubits and the edges represent interactions between the qubits. This specificity of the quantum hardware is based on the fact that both H_I and H_P are restricted in AQC to be of quadratic form, that is,

$$\sum_{i\in V}\alpha_i X_i + \sum_{(i,j)\in E}\beta_{(i,j)}X_iX_j \tag{2}$$

where E and V are respectively the set of edges and vertices of the *program graph*, Γ_P, that can be constructed from $H(t)$ by viewing it as a weighted adjacency matrix for Γ_P. Furthermore, α_i and $\beta_{(i,j)}$ are elements of the set $\{0, 1\}$. Compiling an AQC program is the process of mapping Γ_P into Γ_H, where the variables X_i are taken to be the qubits in the hardware graph and the binary nature of α_i and $\beta_{(i,j)}$ is used to specify whether the qubits X_i and X_j are in use or not.

A. Compiling AQC programs

For the set of vertices and edges V_P, E_P respectively of Γ_P, and the set of vertices and edges V_H, E_H respectively of Γ_H, we make the following definitions.

Definition 1: An *ideal compilation* is a function $f : V_P \to V_H$ such that if $(u, v) \in E_P$ then $(f(u), f(v)) \in E_H$.

Definition 2: A *non-ideal compilation* is a relation $r : \Gamma_P \to \Gamma_H$ such that

i) for $v \in V_P$, $r(v)$ is the vertex set of a connected subtree T_v of Γ_H.

ii) for $(u, v) \in E_P$, there exists $i_u, i_v \in V_H$ such that $i_u \in T_u$, $i_v \in T_v$, and $(i_u, i_v) \in E_H$.

An ideal compilation of an AQC program is obviously one for which the hardware graph has the smallest number of edges for the quantum processor to process. We consider this property to be a contribution toward saving processing time. There can be any number of non-ideal compilations of the program, but our goal here is to identify one which is optimal in the sense of Nash equilibrium in a network creation game given in [3]. The authors of this paper consider a given finite set of vertices as the set of players whose strategic choices are to identify an optimal subset of the players, so as to maximize connectivity between the players while minimizing the cost incurred in laying edges to establish the connectivity. We will consider a variation of this game in section III. First however, we review some essential game theory in the following section.

B. Non-cooperative Games and Nash Equilibrium

We start by defining a non-cooperative game with finitely many player.

Definition 3: A N-player non-cooperative game is a function

$$G : \prod_{i=1}^{N} S_i \longrightarrow O, \tag{3}$$

with the feature of non-identical preferences defined over the elements of the set of *outcomes* O, for every "player" of the game. The preferences are a pre-ordering of the elements of O, that is, for $l, m, n \in O$

$$m \preceq m, \text{ and } l \preceq m \text{ and } m \preceq n \implies l \preceq n. \tag{4}$$

where the symbol $\preceq$ denotes "of less or equal preference".

Preferences are typically quantified numerically for the ease of calculation of the payoffs. To this end, functions G_i are introduced which act as the *payoff function* for each player i and typically map elements of O into the real numbers in a way that preserves the preferences of the players. That is, $\preceq$ is replaced with $\leq$ when analyzing the payoffs. The factor S_i in the domain of G is said to be the *strategy set* of player i, and a *play* of G is an n-tuple of strategies, one per player, producing a payoff to each player in terms of his preferences over the elements of O in the image of Γ.

Definition 4: (Nash Equilibrium) [5] A play of G in which every player employs a strategy that is a best reply, with respects to his preferences over the outcomes, to the strategic choice of every other player.

In other words, unilateral deviation from a Nash equilibrium by any one player in the form of a different choice of strategy will produce an outcome which is less preferred by that player than before. Following Nash, we say that a play p' of G *counters* another play p if $G_i(p') \geq G_i(p)$ for all players i, and that a self-countering play is an (Nash) equilibrium.

Let C_p denote the set of all the plays of G that counter p. Denote $\prod_{i=1}^{N} S_i$ by S for notational convenience, and note that $C_p \subset S$ and therefore $C_p \in 2^S$. Further note that the game G can be factored as

$$G : S \xrightarrow{G_C} 2^S \xrightarrow{E} O \tag{5}$$

where to any play p the map G_C associates its countering set C_p via the payoff functions G_i. The set-valued map G_C may be viewed as a preprocessing stage where players seek out a self-countering play, and if one is found, it is mapped to its corresponding outcome in O by the function E. The condition for the existence of a self-countering play, and therefore of a Nash equilibrium, is that G_C have a fixed point, that is, an element $p^* \in S$ such that $p^* \in G_C(p^*) = C_{p^*}$.

In a general set-theoretic setting for non-cooperative games, the map G_C may not have a fixed point. Hence, not all non-cooperative games will have a Nash equilibrium. However, according to Nash's theorem, when the S_i are finite and the game is extended to its *mixed* version, that is, the version in which randomization via probability distributions is allowed over the elements of all the S_i, as well as over the elements of O, then G_C has at least one fixed point and therefore at least one Nash equilibrium.

Formally, given a game G with finite S_i for all i, its mixed version is the product function

$$\Lambda : \prod_{i=1}^{N} \Delta(S_i) \longrightarrow \Delta(O) \tag{6}$$

where $\Delta(S_i)$ is the set of probability distributions over the i^{th} player's strategy set S_i, and the set $\Delta(O)$ is the set of probability distributions over the outcomes O. Payoffs are now calculated as *expected payoffs*, that is, weighted averages of the values of G_i, for each player i, with respect to probability distributions in $\Delta(O)$ that arise as the product of the plays of Λ. Denote the expected payoff to player i by the function Λ_i. Also, note that Λ restricts to Γ.

In these games, at least one Nash equilibrium play is guaranteed to exist as a fixed point of Λ via Kakutani's fixed-point theorem.

Theorem 1: (Kakutani fixed-point theorem) [6] *Let $S \subset \mathbb{R}^n$ be nonempty, compact, and convex, and let $\Gamma : S \to 2^S$ be an upper semi-continuous set-valued mapping such that $\Gamma(s)$ is non-empty, closed, and convex for all $s \in S$. Then there exists some $s^* \in S$ such that $s^* \in \Gamma(s^*)$.*

To be more specific, set $S = \prod_{i=1}^{N} \Delta(S_i)$. Then $S \subset \mathbb{R}^n$ and S is non-empty, bounded, and closed because it is a finite product of finite non-empty sets. The set S is also convex because it is the Cartesian product of the convex sets $\Delta(S_i)$. Next, let C_p be the set of all plays of Λ that counter the play p. Then C_p is non-empty, closed, and convex. Further, $C_p \subset S$ and therefore $C_p \in 2^S$. Since Λ is a game, it factors according to (5)

$$\Lambda : S \xrightarrow{\Lambda_C} 2^S \xrightarrow{E_\Pi} \Delta(O) \tag{7}$$

where the map Λ_C associates a play to its countering set via the payoff functions Λ_i. Since Λ_i are all continuous, Λ_C is continuous. Further, $\Lambda_C(s)$ is non-empty, closed, and convex for all $s \in S$ (we invite the reader to check that the convexity of C_p is immediate when the payoff functions are linear). Hence, Kakutani's theorem applies and there exists an $s^* \in S$ that counters itself, that is, $s^* \in \Lambda_C(s^*)$, and is therefore a Nash equilibrium. The function E_Π simply maps s^* to $\Delta(O)$ as the product probability distribution from which the Nash equilibrium expected payoff is computed for each player.

III. Non-ideal Compilation as a Non-cooperative Game

We wish to interpret the non-ideal compilation problem as a type of network creation game, and use Kakutani fixed-point stability to show that a compilation at Nash equilibrium exists Recall that in this game, the players choose any subset of vertices they wish to connect to. They can choose to build connections with all the players, with some of the players, or with none of them. However, if at the end of the game, a player is not connected to all the others, he/she is penalized. This is why even though it would cost a player to build a connection, in some cases, not building one could cost more.

In the compilation game on the other hand, a player's strategy is to choose a subset of vertices in Γ_H such that a smallest possible subtree of Γ_H spans this choice while keeping Definition 2 true. This is because the penalty to the players is defined in terms of the number of edges (which is directly proportional to the number of vertices) in the subtree it is mapped to.

Formally, the *non-ideal compilation game* is the function

$$C : S = \prod_{i=1}^{n} S_i \to 2^S \to \Gamma_H$$

where the countering sets are calculated using the cost (payoff) function

$$G_i(s) = \alpha \cdot |S_i| \tag{8}$$

for player i, when strategy $s \in S_i$ is employed, and where the cost of laying down one edge is α. Note that G_i are linear and hence the countering sets will be convex. However, for Nash equilibrium guarantee via Kakutani's fixed-point theorem, we also need to ensure that the set of plays $S = (S_1, S_2, \ldots, S_n)$ of the game is also compact and convex in the topological space that it resides in, that is, $\mathbb{R}^2$.

Since each S_i is a a tree inside $\Gamma_H \subset \mathbb{R}^2$, it is bounded; it is also closed in $\mathbb{R}$ with respect to the relative topology and hence closed in $\mathbb{R}^2$. Therefore, each S_i is compact in $\mathbb{R}^2$ and hence so is $S = \prod_{i=1}^{n} S_i$. The set S is convex only if each S_i is, or in other words, each S_i is a line segment or a single edge tree, in which case the Kakutani fixed-point theorem applies and a Nash equilibrium exists in the game C.

To consider a larger class of pure strategies in the game C for which Nash equilibrium may be guaranteed, extend S to its convex hull $\mathrm{Conv}(S)$ which is compact by Caratheodory's theorem [4]. We now have fixed points for set-valued functions

$$F : \mathrm{Conv}(S) \to 2^{\mathrm{Conv}(S)}.$$

It is a basic fact in topology that any compact, convex subset of $\mathbb{R}^m$ is homeomorphic to a closed ball $\mathbb{B}^m$ for all m. We therefore have that $\mathrm{Conv}(S) \cong \mathbb{B}^2$ and

$$F : \mathbb{B}^2 \to 2^{\mathbb{B}^2}.$$

To get fixed points for functions

$$\mathcal{F} : S \to 2^S$$

using F, construct a retract

$$R : \mathbb{B}^2 \to S.$$

By definition, a retract has the property that $R(v) = v$ for all $v \in S \subset \mathbb{B}^2$. Note that

$$\mathcal{F}(v) = F(R(v))$$

so that the fixed points of $\mathcal{F}$ lie in S.

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