A Revised Formulation of Maxwell's Equations

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***Abstract*—In 1864, James Clerk Maxwell formulated a set of electromagnetic equations to describe the interactions between electric and magnetic fields, now known as Maxwell's equations in his honor. In this paper, using a simple thought experiment, it will be proven that Faraday's law must be revised to correctly describe the relation of the electric field generated from a time-varying magnetic field in a medium, whose relative permittivity is not unity. Ampere's law will also be modified to reflect the change in Faraday's law. Additional changes will be made to adhere the equations to the law of conservation of energy. The new set of electromagnetic equations not only does not conflict with the existing formulation, but also includes a previously unaccounted relation of a non-Ohmic current, and its generation of the magnetic field.**

***Index Terms*—Maxwell's equations, electric field, magnetic field, electric-flux density, magnetic-flux density, Faraday's law, Ampere's law**

I. Introduction

JAMES Clerk Maxwell published 3 famous papers on electromagnetism [1] – [3]. He translated Faraday's experiment results into a mathematical equation, known as Faraday's law today, and introduced the displacement current. In his famous 1864 paper [3], he presented a theory of the electromagnetic field, including his hypothesis of the existence of electromagnetic waves, which was confirmed by Heinrich Hertz in 1893 [4]. Maxwell did make use of mechanical analogies to discover the formulation of electromagnetic equations. An example of the derivation of Faraday's law will be presented.

In mechanics, the relation between the change in momentum $\Delta\vec{p}$, and the impulse caused by a force $\vec{F}$ is

$$\Delta\vec{p} = \int_T \vec{F}\,dt, \tag{1}$$

where T is the time interval over which the integration is carried out, and $\Delta\vec{p}$ is the change in momentum during this time interval. The electric field generated by a changing magnetic flux across a loop area, was viewed as a "force" and an impulse delivered to the loop, resulting in a change in momentum. The electric field $\vec{E}$ was even referred to as the electromotive *force*. If $\vec{E}$ is the electric field generated by a time-varying magnetic flux, from the relation between impulse and momentum,

$$\vec{A} = \int_T \vec{E}\,dt, \tag{2}$$

and $\vec{A}$ was called the electromagnetic momentum. Today $\vec{A}$ is known as the magnetic vector potential. Using the above equation, the electric field can be written as

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t}. \tag{3}$$

Although Faraday's law can be formulated without the negative sign in the above equation, the reader is referred to [5] for more details on the origin of the negative sign. The total momentum of a loop ℓ was calculated as

$$\oint_\ell \vec{A}\cdot\vec{dl}. \tag{4}$$

Using the Kelvin-Stokes theorem, the path integral can be written as an area integral over S, that is enclosing the loop,

$$\oint_\ell \vec{A}\cdot\vec{dl} = \int_S \left(\nabla\times\vec{A}\right)\cdot\vec{dA}. \tag{5}$$

If the right-hand side of the above equation is set equal to the changing magnetic-flux across the loop area,

$$\int_S \left(\nabla\times\vec{A}\right)\cdot\vec{dA} = \int_S \vec{B}\cdot\vec{dA}, \tag{6}$$

the above equation can be viewed as a conservation of momentum equation, where the "momentum" of the changing magnetic flux is set equal to the "momentum" of the impulse delivered to the loop by the electric field. If the above equation is satisfied for any loop, then

$$\vec{B} = \nabla\times\vec{A}. \tag{7}$$

Faraday's law was written as a set of two equations: Equation 3 and Equation 7. Applying the $\nabla\times$ operation on both sides of Equation 3, and substituting Equation 7,

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{8}$$

the resulting equation is the same as how Faraday's law is written today, as a single equation. There is no justification for the use of $\vec{E}$ in the above equation, instead of the electric-displacement vector $\vec{D}$, which is called electric-flux density today. It will be shown by a very simple thought experiment, Faraday's law written as

$$\nabla\times\vec{D} = -\frac{\partial\vec{B}}{\partial t}, \tag{9}$$

is the correct formulation.

This paper requires a good understanding of the differences between electric field and electric-flux density. This will be reviewed first in Section II. Maxwell's equations in electrostatic units (ESU) will be summarized in Section III. An inaccuracy in the formulation of Faraday's law will be explained by a simple thought experiment in Section IV, followed by the present view of $\vec{E}$ and $\vec{D}$ in Maxwell's equations in Section V. Additional changes resulting from the modification to Faraday's law will be presented in Section VI – Section VIII. The derivation of the current continuity equation in the revised formulation is presented in

Section IX. A summary of the revised Maxwell's equations is presented in Section X, followed by a note on the definition of the displacement current in the revised formulation in Section XI. The derivation of the wave equation in the revised formulation will be presented in Section XII, to show that the new formulation does not conflict with the existing set of Maxwell's equations. Other possibilities in the formulation of the modified Ampere's law are incorrect, as explained in Section XIII. The revised Ohm's law is presented in Section XIV. The new set of equations in electrostatics and magnetostatics will be presented in Section XV – Section XVI.

II. Electric Field $\vec{E}$ vs Electric-Flux Density $\vec{D}$

In this paper, Maxwell's equations will be written in electrostatic units (ESU) [5]. In this system of CGS units, electric field $\vec{E}$ and electric-flux density $\vec{D}$ have the same units [5] [6],

$$[\vec{E}] = [\vec{D}] = \frac{gm^{1/2}}{cm^{1/2}\cdot sec}. \tag{10}$$

This is a useful property for the new set of equations formulated, as explained in Section IV, since $\vec{E}$ can be replaced by $\vec{D}$ without requiring modifications to any of the remaining units. ESU reminds us that although electric field $\vec{E}$ and electric-flux density $\vec{D}$ are used as a path integral in Faraday's law, and area integral in Gauss's law respectively, both are *electric fields*. Although the new set of equations will be formulated in ESU, it can also be derived in other systems of units, including the present SI system.

Faraday's experiment with spherical capacitors is presented first to highlight the differences between electric field $\vec{E}$ and electric-flux density $\vec{D}$. Faraday used two identical spherical capacitors A and B, whose cross sections are shown in Figure 1 [5]. The conductors of A are marked p and q, and the conductors of B, p' and q'. The outer conductors of A and B, and the inner conductors, are connected together, so that the potential differences V_{pq} and $V_{p'q'}$ are equal.

The cavity of B is filled with a dielectric material shown shaded. When both the capacitors are charged, for example, using a Wimshurst machine, B stores $\epsilon_r\times$ more charge than A. This can be measured by discharging each of the capacitors through a ballistic galvanometer. This has been explained in great detail in Reference [5], including all the mathematical derivations.

This result can be explained as follows: the dielectric material reduces the strength of the electric field by the factor ϵ_r, known as the permittivity of the material. There is $\epsilon_r\times$ more charge present in B, to compensate for the reduction in the electric field, so that the voltages of each of the capacitors are equal. If the electric field $\vec{E}$ is unmodified by the dielectric material, the electric field would have been the electric-flux density $\vec{D}$. This is written in ESU as

$$\vec{D} = \epsilon_r \vec{E}, \tag{11}$$

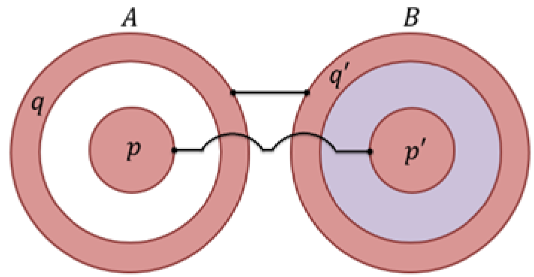

Fig. 1. Setup of Faraday's experiment with spherical capacitors [5].

where ϵ_r is the relative permittivity of the material. $\vec{D}$ can be viewed as the applied field in a material, and the material reduces the strength to $\vec{E}$. The above equation may also written as a sum of the fields,

$$\vec{D} = \vec{E} + 4\pi\vec{P}, \tag{12}$$

where $\vec{P}$ is the polarization vector. The polarization vector is written in terms of the electric susceptibility χ_e as

$$\vec{P} = \chi_e \vec{E}. \tag{13}$$

From the above equations,

$$\epsilon_r = 1 + 4\pi\chi_e. \tag{14}$$

From Faraday's experiments with spherical capacitors, it can be noted that $\vec{D}$ is also an electric field. It is the electric field at any point, without including the effect of a material on the field at that point.

III. Maxwell's Equations in Electrostatic Units

Maxwell's equations in the differential form in ESU is summarized in this section [5]. Gauss's law in ESU satisfies the relation,

$$\nabla\cdot\vec{D} = 4\pi\rho, \tag{15}$$

where ρ is volume-charge density, $\vec{D}$ is the electric-flux density. Ampere's law is

$$\nabla\times\vec{H} = 4\pi\left(\vec{J}_O + \frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}\right), \tag{16}$$

where $\vec{J}_O$ is conduction current, which is the current resulting from the flow of electric charges, and $\vec{H}$ is the magnetic field. The term added to $\vec{J}_O$ in the above equation is the displacement current term,

$$\vec{J}_d = \frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}. \tag{17}$$

This term satisfies the current continuity equation across the dielectric material of a parallel-plate capacitor [5]. Faraday's law is

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{18}$$

where $\vec{E}$ is the electric field generated by a time-varying magnetic-flux density $\vec{B}$. The divergence-free condition of $\vec{B}$ is

$$\nabla\cdot\vec{B} = 0. \tag{19}$$

In ESU, the relation between magnetic-flux density $\vec{B}$ and magnetic field $\vec{H}$ is

$$\vec{B} = \frac{\mu_r}{c^2}\vec{H}, \tag{20}$$

where c is the speed of light, and μ_r is the relative permeability. The relation between $\vec{D}$ and $\vec{E}$ is written in Equation 11.

IV. An Inconsistency in the Formulation of Faraday's Law

A very simple thought experiment is presented in this section to show that the formulation of Faraday's law is inconsistent, considering the behavior of electric-flux density $\vec{D}$ and electric field $\vec{E}$ in a dielectric material.

Two cases are analyzed: the material in Case 1 in Figure 2 is a uniform dielectric material of relative permittivity ϵ_{r1}, while the region in Case 2 is filled with a material of permittivity ϵ_{r2}. Case 1 and Case 2 are identical, except for the permittivity of the material that fills the space. The time-varying magnetic-flux density $\vec{B}$ at all the points are equal in both the cases.

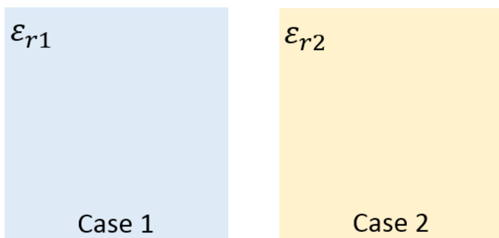

Fig. 2. Two media with relative permittivity ϵ_{r1} and ϵ_{r2}.

From Faraday's law in Equation 8, since the rate of change of magnetic flux are equal in both the cases, the electric field $\vec{E}$ generated in each of the cases must be equal as well. The electric-flux density, however, $\vec{D}_1$ in Case 1, and $\vec{D}_2$ in Case 2, are different,

$$\vec{D}_1 = \epsilon_{r1}\vec{E} \tag{21}$$

$$\vec{D}_2 = \epsilon_{r2}\vec{E} \neq \vec{D}_1, \tag{22}$$

since the material properties are different.

A closer analysis of $\{\vec{D}_1,\ \vec{E}\}$ and $\{\vec{D}_2,\ \vec{E}\}$ shows that the field values are non-physical. The final electric field in the material can be viewed as a two-step process: in Step 1, $\vec{D}$ by definition, is the electric field that is unmodified by a dielectric material, and must have been the field generated by time-varying $\vec{B}$. In Step 2, the material reduces the strength of $\vec{D}$ by ϵ_r, resulting in $\vec{E}$.

Since $\vec{B}$ is the same in both the cases, the value of $\vec{D}$ generated cannot be different in both the cases, unlike Equation 22. The material in Case 1 modifies $\vec{D}$ different from Case 2, and the electric fields $\vec{E}$ may be different. In the present formulation of Faraday's law, this is reversed: $\vec{E}$ are equal, but $\vec{D}$ are different in the two cases.

One may argue that the analysis of the thought experiment must include the coupling effect between Faraday's law and Ampere's law. However, each of the Maxwell's equations must be well formulated by itself, even without any coupling between the equations. Alternately, if $\vec{B}$ varies linearly over time, from Equation 18, the electric field generated must be a constant, and there is no coupling between Equation 16 and Equation 18, Ampere's law and Faraday's law. The motivation to modify Faraday's law would still be applicable.

This non-physical result can be fixed by formulating Faraday's law as

$$\nabla \times \vec{D} = -\frac{\partial \vec{B}}{\partial t}. \tag{23}$$

From the above equation, $\vec{D}$ generated in both the cases are equal. The material modifies $\vec{D}$ differently. Using Equation 11, the electric field $\vec{E}_1$ and $\vec{E}_2$ can be calculated as

$$\vec{E}_1 = \frac{\vec{D}}{\epsilon_{r1}} \tag{24}$$

$$\vec{E}_2 = \frac{\vec{D}}{\epsilon_{r2}} \neq \vec{E}_1, \tag{25}$$

and the results are now in agreement with the physical behavior of electric field in a dielectric material.

Electric field $\vec{E}$ and electric-flux density $\vec{D}$ have the same units in ESU. This system of units has the advantage of swapping $\vec{E}$ by $\vec{D}$, and not affecting the units of the electrical quantities. This is the reason for choosing to write the electromagnetic equations in this paper in ESU. However, the revised set of equations presented in this paper can be formulated in any system of units, including SI.

V. Existing View of $\vec{E}$ and $\vec{D}$ in Maxwell's Equations

In the existing formulation of Maxwell's equations, the relation between $\vec{D}$ and $\vec{E}$ is treated similar to the relation between $\vec{B}$ and $\vec{H}$, the magnetic-flux density and the magnetic field, respectively. The presence of $\vec{H}$ in a magnetic material, modifies the field to $\vec{B}$. Likewise, since the electric fields generated are equal in both the cases in the thought experiment, Maxwell's equations view $\vec{E}$ as the field generated, and the material modifies this field to $\vec{D}$, similar to the definitions of $\vec{B}$ and $\vec{H}$. However, as seen in Faraday's experiment with spherical capacitors in Section II, this is not the physical behavior of $\vec{E}$ in a dielectric material. $\vec{D}$ is the field that would exist, if a dielectric material does not modify the field. $\vec{E}$ is the field including the effect of the dielectric material.

VI. Modifying Ampere's Law

If Faraday's law in Equation 18 is modified as written in Equation 23, then Ampere's law must also be modified accordingly, to be able to derive the wave equation. To obtain the wave equation, Ampere's law must be written as

$$\nabla \times \vec{H} = \epsilon_r \left(4\pi \vec{J}_O + \frac{\partial \vec{D}}{\partial t} \right). \tag{26}$$

In the above equation,

$$\vec{J}_O = \sigma \vec{E} \tag{27}$$

is the volume current density in the case when $\epsilon_r = 1$. The above equation is the same as Ohm's law written using fields [5]. $\vec{J}_O$ scales by ϵ_r, when the current flows in a material with permittivity ϵ_r. This has some similarity to Faraday's experiment results with spherical capacitors in Figure 1, where there is $\epsilon_r\times$ more charge in the capacitor whose cavity is filled with a dielectric material.

Other possibilities in the formulation of Ampere's law will be considered in Section XIII, but it will be proven that the other formulations are incorrect. Additional modifications to the electromagnetic equations will be done in the following sections. The wave equation from the revised set of electromagnetic equations will be derived in Section XII.

Note that it would be incorrect to write the left-hand side of Equation 26 as

$$\nabla \times \vec{B} \tag{28}$$

instead of

$$\nabla \times \vec{H}. \tag{29}$$

This will be explained in the remainder of this section.

Considering only the conduction current portion of Equation 26,

$$\nabla \times \vec{H} = 4\pi\epsilon_r \vec{J}_O. \tag{30}$$

If Equation 30 is incorrectly written as

$$\nabla \times \vec{B} = 4\pi\epsilon_r \vec{J}_O, \tag{31}$$

substituting Equation 20, and assuming a uniform magnetic material with permeability μ_r,

$$\nabla \times \vec{H} = 4\pi c^2 \left(\frac{\epsilon_r}{\mu_r}\right) \vec{J}_O. \tag{32}$$

If the above equation is applied on two different cases, with two different uniform permeability values, but the same volume current density and permittivity, two different values of $\nabla \times \vec{H}$ are obtained. Since the magnetic field $\vec{H}$ is the field, without the effect of magnetization in a material, the same volume current density generates two different values of $\nabla \times \vec{H}$, which is clearly an incorrect result. Therefore, $\nabla \times \vec{H}$ in Equation 26 is the correct formulation.

VII. Additional Revisions

The revised set of electromagnetic equations in ESU are

$$\nabla \cdot \vec{D} = 4\pi\rho \tag{33}$$

$$\nabla \cdot \vec{B} = 0 \tag{34}$$

$$\nabla \times \vec{D} = -\frac{\partial \vec{B}}{\partial t} \tag{35}$$

$$\nabla \times \vec{H} = \epsilon_r \left(4\pi \vec{J}_O + \frac{\partial \vec{D}}{\partial t}\right). \tag{36}$$

In the electrostatic case, there are no time-varying fields, and the right-hand side of Equation 35 reduces to 0,

$$\nabla \times \vec{D} = 0, \tag{37}$$

where $\vec{D}$ is a static field.

The conservative property of the electric field $\vec{E}$ must be met in electrostatics [5],

$$\nabla \times \vec{E} = 0, \tag{38}$$

and not $\vec{D}$. In the presence of a dielectric material, the field that is present is $\vec{E}$, and not $\vec{D}$. Not meeting the above equation is a violation of the law of conservation of energy.

Additional revisions are needed, so that

$$\nabla \times \vec{E} = 0, \tag{39}$$

is satisfied in the electrostatic case.

VIII. Existing Formulation of Maxwell's Equations Using $\{\vec{D}_C, \vec{D}_F\}$ and $\{\vec{E}_C, \vec{E}_F\}$

The law of conservation of energy will be enforced by writing $\vec{D}$ and $\vec{E}$ as the superposition of two fields, depending on the source of the field: from electric charges, or Faraday's law. The existing form of Maxwell's equations will be written using superposition of the two fields. This will be mathematically derived next. It will become clear in Section X, how this would make the equations adhere to the law of conservation of energy.

The magnetic vector potential $\vec{A}$ is defined as

$$\vec{B} = \nabla \times \vec{A}. \tag{40}$$

Substituting the above equation in the present formulation of Faraday's law,

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right). \tag{41}$$

Rearranging the above equation,

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{42}$$

From calculus, if the curl of a vector field is 0, the vector field can be written as the gradient of a scalar field,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\Phi. \tag{43}$$

Rearranging the above equation,

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{44}$$

$$= \vec{E}_C + \vec{E}_F. \tag{45}$$

From the above equation, $\vec{E}$ can be written as the sum of a Coulomb field,

$$\vec{E}_C = -\nabla\Phi, \tag{46}$$

and the electric field generated from Faraday's law,

$$\vec{E}_F = -\frac{\partial \vec{A}}{\partial t}. \tag{47}$$

$\vec{E}_C$ is the electric field generated by electric charges. This can be proven mathematically, using Coulomb gauge,

$$\nabla \cdot \vec{A} = 0, \tag{48}$$

to show that $\vec{E}_C$ is the instantaneous Coulomb field [7] [8]. $\vec{E}_F$ is the electric field generated from Faraday's law. Applying the $\nabla\times$ operator on both sides of Equation 47, and substituting Equation 40,

$$\nabla \times \vec{E}_F = -\frac{\partial \vec{B}}{\partial t}, \tag{49}$$

which is the same as Faraday's law.

Since $\vec{E}_C$ is the gradient of a scalar potential in Equation 46, from calculus,

$$\nabla \times \vec{E}_C = 0. \tag{50}$$

Similar to Equation 45, $\vec{D}$ at any point can be written as

$$\vec{D} = \vec{D}_C + \vec{D}_F. \tag{51}$$

At any point, Equation 15 must be satisfied. Substituting Equation 51 in Equation 15,

$$\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F = 4\pi\rho, \tag{52}$$

is Gauss's law. Using Equation 51, Ampere's law is written as

$$\nabla \times \vec{H} = 4\pi \vec{J}_O + \frac{\partial \vec{D}_C}{\partial t} + \frac{\partial \vec{D}_F}{\partial t}. \tag{53}$$

Applying the $\nabla\cdot$ operation on both the sides of the above equation,

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = 4\pi \nabla \cdot \vec{J}_O + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_C\right) + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_F\right). \tag{54}$$

From calculus, the left-hand side reduces to 0, since $\nabla\cdot\nabla\times$ of a vector function is 0. Using Equation 52, Equation 54 reduces to the current continuity equation,

$$\nabla \cdot \vec{J}_O = -\frac{\partial \rho}{\partial t}. \tag{55}$$

From the above results, the existing set of Maxwell's equations in ESU can be written as

$$\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F = 4\pi\rho \tag{56}$$

$$\nabla \cdot \vec{B} = 0 \tag{57}$$

$$\nabla \times \vec{E}_F = -\frac{\partial \vec{B}}{\partial t} \tag{58}$$

$$\nabla \times \vec{E}_C = \vec{0} \tag{59}$$

$$\nabla \times \vec{H} = 4\pi \vec{J}_O + \frac{\partial \vec{D}_C}{\partial t} + \frac{\partial \vec{D}_F}{\partial t}, \tag{60}$$

and the above equations are also valid for time-varying fields [5]. In addition to the above equations,

$$\vec{D}_F = \epsilon_r \vec{E}_F \tag{61}$$

$$\vec{D}_C = \epsilon_r \vec{E}_C \tag{62}$$

$$\vec{B} = \mu_o \mu_r \vec{H} \tag{63}$$

$$\mu_o = \frac{1}{c^2}, c \approx 3.0 \times 10^{10}\, cm/s, \tag{64}$$

capture the effect of a material on the fields.

IX. Current Continuity Equation in the Revised Formulation

The revised formulation of Ampere's law, written using $\{\vec{D}_C, \vec{D}_F\}$ is

$$\nabla \times \vec{H} = \epsilon_r \left(4\pi \vec{J}_O + \frac{\partial \vec{D}_C}{\partial t} + \frac{\partial \vec{D}_F}{\partial t}\right), \tag{65}$$

obtained by substituting Equation 51 in Equation 26. The current continuity equation from the present form of Ampere's law was derived in Equation 55. In the new formulation, not one, but two current continuity equations will be derived.

Applying the $\nabla\cdot$ operation on both sides of the above equation,

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \left(4\pi\epsilon_r \vec{J}_O\right) + \frac{\partial}{\partial t}\nabla \cdot \left(\epsilon_r \vec{D}_C\right) + \frac{\partial}{\partial t}\nabla \cdot \left(\epsilon_r \vec{D}_F\right). \tag{66}$$

From calculus, the left-hand side of Equation 66 reduces to 0. Using Equation 14, the above equation can be written as

$$0 = \left[4\pi\nabla \cdot \left(\vec{J}_O\right) + \frac{\partial}{\partial t}\nabla \cdot \left(\vec{D}_C + \vec{D}_F\right)\right] + \left[4\pi\nabla \cdot \left(4\pi\chi_e \vec{J}_O\right) + 4\pi\frac{\partial}{\partial t}\nabla \cdot \left(\chi_e \vec{D}_C + \chi_e \vec{D}_F\right)\right]. \tag{67}$$

Setting the expressions within the two square brackets to be 0, two current continuity equations can be obtained. Setting the first expression to 0,

$$4\pi\nabla \cdot \vec{J}_O + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_C\right) = 0, \tag{68}$$

and substituting Equation 52,

$$\nabla \cdot \vec{J}_O = -\frac{\partial \rho}{\partial t}, \tag{69}$$

is the familiar current continuity equation. Setting the second expression within the square brackets in Equation 67 to 0,

$$4\pi\nabla \cdot \left(4\pi\chi_e \vec{J}_O\right) + 4\pi\frac{\partial}{\partial t}\nabla \cdot \left(\chi_e \vec{D}_C + \chi_e \vec{D}_F\right) = 0. \tag{70}$$

This equation is the current continuity equation for

$$J_P = 4\pi\chi_e \vec{J}_O, \tag{71}$$

which will be referred to as the non-Ohmic current, as opposed to the Ohmic current in Equation 27. In the case where χ_e is a constant, this term can be moved out of the $\nabla\cdot$ operator,

$$\nabla \cdot \left(4\pi\chi_e \vec{J}_O\right) + \chi_e \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F\right) = 0. \tag{72}$$

The χ_e term can be factored and cancelled. Substituting Equation 52, the same current continuity equation as

Equation 68 is obtained. This shows that Equation 70 is the current continuity equation for $\vec{J}_P$, in general, when χ_e is not a constant.

One of the current continuity equations must be solved together with the remaining field equations. If one of the current continuity equations is satisfied, the second continuity equation is automatically satisfied. This can be seen from Equation 67. More details will be discussed in Section X.

X. Revised Formulation of Maxwell's Equations

The revised form of Maxwell's equations is written in terms of $\{\vec{D}_C, \vec{D}_F\}$, and $\{\vec{E}_C, \vec{E}_F\}$, so that the equations adhere to the law of conservation of energy. This will become clear in this section, after the revised set of equations are presented.

The derivation of the two current continuity equations is a logical proof of the validity of Gauss's law in the revised formulation,

$$\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F = 4\pi\rho. \tag{73}$$

In Equation 45, the total electric field is written as a superposition of the Coulomb field and the field resulting from Faraday's law. Since, only Faraday's law is modified, the conservative property of $\vec{E}_C$,

$$\nabla \times \vec{E}_C = \vec{0}, \tag{74}$$

will be kept unmodified.

The existing formulation of Faraday's law in Equation 49 is modified as

$$\nabla \times \vec{D}_F = -\frac{\partial \vec{B}}{\partial t}, \tag{75}$$

as explained in Section IV. The divergence-free condition of $\vec{B}$

$$\nabla \cdot \vec{B} = 0, \tag{76}$$

can be derived from Equation 75, using a similar derivation in [5]. Applying the $\nabla\cdot$ operation on both sides of Equation 75,

$$\nabla \cdot \left(\nabla \times \vec{D}_F\right) = -\frac{\partial}{\partial t}\left(\nabla \cdot \vec{B}\right). \tag{77}$$

The left-hand side reduces to 0,

$$0 = -\frac{\partial}{\partial t}\left(\nabla \cdot \vec{B}\right). \tag{78}$$

The above equation is satisfied if $\nabla \cdot \vec{B}$ is a constant that does not vary over time. Using a proof by contradiction, it will be shown that Equation 76 must be satisfied. Lets assume that $\nabla \cdot \vec{B}$ is a non-zero value. If the source of $\vec{B}$ decays to 0, or is moved far away so that $\vec{B}$ decays to 0, then $\nabla \cdot \vec{B}$ is 0. This contradicts the initial assumption that $\nabla \cdot \vec{B}$ is a non-zero value. Therefore, the only possible value for $\nabla \cdot \vec{B}$ is 0.

In the existing Maxwell's equations, if Gauss's law is satisfied in the solution of fields, then the current continuity equation in Equation 55 is automatically satisfied. However, in the revised formulation of Maxwell's equations, this is not the case. Although Gauss's law may be satisfied, there are two current continuity equations that needs to be satisfied in Equation 68 and Equation 70. From Equation 67, if either Equation 68 or Equation 70 is satisfied, the other current continuity equation is automatically satisfied. Therefore, one of the current continuity-equations needs to be solved with the remaining field equations.

The revised set of Maxwell's equations are

$$\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F = 4\pi\rho \tag{79}$$

$$\nabla \cdot \vec{B} = 0 \tag{80}$$

$$\nabla \times \vec{D}_F = -\frac{\partial \vec{B}}{\partial t} \tag{81}$$

$$\nabla \times \vec{E}_C = \vec{0} \tag{82}$$

$$\nabla \times \vec{H} = \epsilon_r\left(4\pi\vec{J}_O + \frac{\partial \vec{D}_C}{\partial t} + \frac{\partial \vec{D}_F}{\partial t}\right), \tag{83}$$

and either the first current continuity equation,

$$4\pi\nabla \cdot \vec{J}_O + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F\right) = 0, \tag{84}$$

or the second continuity equation,

$$4\pi\nabla \cdot \left(4\pi\chi_e\vec{J}_O\right) + 4\pi\frac{\partial}{\partial t}\left(\nabla \cdot \left(\chi_e\vec{D}_C + \chi_e\vec{D}_F\right)\right) = 0 \tag{85}$$

needs to be solved together with Equation 79 – Equation 83. Repeating Equation 61 – Equation 64,

$$\vec{D}_F = \epsilon_r\vec{E}_F \tag{86}$$

$$\vec{D}_C = \epsilon_r\vec{E}_C \tag{87}$$

$$\vec{B} = \mu_o\mu_r\vec{H} \tag{88}$$

$$\mu_o = \frac{1}{c^2}, c \approx 3.0 \times 10^{10}\, cm/s, \tag{89}$$

these equations capture the effect of a material on the fields. The above equations are also valid for time-varying fields and time-varying sources.

As stated earlier, Equation 38 must be satisfied in electrostatics to adhere to the law of conservation of energy. Using Equation 45, $\nabla \times \vec{E}$ at any point is

$$\begin{aligned}\nabla \times \vec{E} &= \nabla \times \left(\vec{E}_C + \vec{E}_F\right)\\ &= \nabla \times \vec{E}_C + \nabla \times \vec{E}_F.\end{aligned} \tag{90}$$

From Equation 82,

$$\nabla \times \vec{E}_C = 0. \tag{91}$$

If there are no time-varying fields in electrostatics, $\vec{E}_F = 0$, since a time-varying magnetic field is required for a non-zero $\vec{E}_F$. Therefore, Equation 90 can be simplified as

$$\nabla \times \vec{E} = 0, \tag{92}$$

thereby adhering to the law of conservation of energy.

XI. Displacement Current

The displacement current term in Maxwell's equations is

$$\vec{J}_d = \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}, \tag{93}$$

and satisfies the "current continuity" across a capacitor dielectric [5].

The derivation that shows the current continuity across a dielectric in a parallel-plate capacitor, assumes an ideal capacitor with no fringing fields. The displacement current, however, is not a real current flow. There is no physical requirement that the displacement current must equal the conduction current in an ideal parallel-plate capacitor. The displacement current, however, may still be defined as written in Equation 93, in the revised formulation.

XII. Derivation of the Wave Equation in the Revised Formulation

In a source-free region, and a uniform medium with permittivity ϵ_r and permeability μ_r, sufficiently far away that $\vec{D}_C$ has decayed to 0, the revised Ampere's law and Faraday's law are

$$\nabla \times \vec{D}_F = -\frac{\partial \vec{B}}{\partial t} \tag{94}$$

$$\nabla \times \vec{H} = \epsilon_r \frac{\partial \vec{D}_F}{\partial t}. \tag{95}$$

In a source-free region, and if $\vec{D}_C$ has decayed to 0, Gauss's law can be simplified to

$$\nabla \cdot \vec{D}_F = 0. \tag{96}$$

In addition, if a uniform material is assumed, Equation 80 can be written as

$$\nabla \cdot \vec{H} = 0. \tag{97}$$

Using the identity,

$$\nabla \times \nabla \times \vec{R} = \nabla\left(\nabla \cdot \vec{R}\right) - \nabla^2 \vec{R}, \tag{98}$$

where $\vec{R}$ is a vector field with the above operations defined,

$$\nabla \times \nabla \times \vec{D}_F = \nabla\left(\nabla \cdot \vec{D}_F\right) - \nabla^2 \vec{D}_F. \tag{99}$$

Simplifying the above equation using Equation 96,

$$\nabla \times \nabla \times \vec{D}_F = -\nabla^2 \vec{D}_F. \tag{100}$$

From Equation 88 – Equation 89 and Equation 94,

$$\nabla \times \nabla \times \vec{D}_F = -\frac{\mu_r}{c^2}\frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right). \tag{101}$$

Since a uniform dielectric medium is assumed, μ_r can be factored out of the $\nabla\times$ operation. Substituting Equation 95 in the above expression,

$$\nabla \times \nabla \times \vec{D}_F = -\frac{\mu_r \epsilon_r}{c^2}\frac{\partial^2 \vec{D}_F}{\partial t^2}. \tag{102}$$

Substituting Equation 100 in Equation 102,

$$\nabla^2 \vec{D}_F = \frac{\mu_r \epsilon_r}{c^2}\frac{\partial^2 \vec{D}_F}{\partial t^2}, \tag{103}$$

is the wave equation. Using Equation 86, the above equation can be written as

$$\nabla^2 \vec{E}_F = \frac{\mu_r \epsilon_r}{c^2}\frac{\partial^2 \vec{E}_F}{\partial t^2}. \tag{104}$$

Similarly, the wave equation of the magnetic field $\vec{H}$ can also be derived.

XIII. Other Possibilities for the Revised Formulation of Ampere's Law

It would be simpler to modify Ampere's law as

$$\nabla \times \vec{H} = 4\pi \vec{J}_O + \frac{\partial \vec{D}_C}{\partial t} + \epsilon_r \frac{\partial \vec{D}_F}{\partial t}, \tag{105}$$

rather than Equation 83. With this formulation, the wave equation can still be derived. The problem with Equation 105 is the lack of symmetry. From Equation 105, $\nabla \times \vec{H}$ is generated by

$$\frac{\partial \vec{D}_C}{\partial t}, \tag{106}$$

but by

$$\epsilon_r \frac{\partial \vec{D}_F}{\partial t}. \tag{107}$$

Both the asymmetrical terms have time-varying electric-flux density in common, but only one of them is scaled by ϵ_r. At any point with relative permittivity ϵ_r, if

$$\frac{\partial \vec{D}_F}{\partial t} = \frac{\partial \vec{D}_C}{\partial t}, \tag{108}$$

this will result in different values of $\nabla \times \vec{H}$, which is clearly non-physical.

Alternately, if Ampere's law is formulated as

$$\nabla \times \vec{H} = 4\pi \vec{J}_O + \epsilon_r \frac{\partial \vec{D}_C}{\partial t} + \epsilon_r \frac{\partial \vec{D}_F}{\partial t}, \tag{109}$$

this results in a meaningless current continuity equation. Substituting Equation 14, applying the $\nabla\cdot$ operation on both sides of the above equation, and simplifying,

$$\begin{aligned} 0 = &\left[4\pi \nabla \cdot \vec{J}_O + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F\right)\right] \\ &+ \left[4\pi \frac{\partial}{\partial t}\nabla \cdot \left(\chi_e \vec{D}_C + \chi_e \vec{D}_F\right)\right]. \end{aligned} \tag{110}$$

Setting the terms in the two square brackets to 0, the first equation is the same as Equation 68, from which the current continuity equation can be derived, as done before in Equation 69. The second equation, however, is

$$4\pi \frac{\partial}{\partial t}\left(\nabla \cdot \left(\chi_e \vec{D}_C + \chi_e \vec{D}_F\right)\right) = 0. \tag{111}$$

Cancelling the 4π, and if χ_e is a constant, the above equation can be simplified as

$$\frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}_C + \nabla \cdot \vec{D}_F\right) = 0. \tag{112}$$

Substituting Equation 79 in the above equation, and simplifying,

$$\frac{\partial \rho}{\partial t} = 0, \tag{113}$$

has no physical meaning, unlike the derivation of the current continuity equation from Equation 72. The only possibility to formulate the revised Ampere's law is Equation 83.

XIV. Revised Ohm's Law

Ohm's law,

$$i = \frac{V}{R}, \tag{114}$$

where i is the current flowing in a wire of resistance R, with voltage V across the wire, can also be written in the differential form as

$$\vec{J}_O = \sigma \vec{E}, \tag{115}$$

where $\vec{J}_O$ is the volume current density resulting from the electric field $\vec{E}$, in a material with conductivity σ [5].

Considering only the conduction current terms from the revised Ampere's law in Equation 83,

$$\nabla \times \vec{H} = 4\pi \epsilon_r \vec{J}_O. \tag{116}$$

Using Equation 14, the above equation can be written as

$$\nabla \times \vec{H} = 4\pi \vec{J}_O + 4\pi \left(4\pi \chi_e \vec{J}_O\right). \tag{117}$$

Substituting Equation 115 in the above equation,

$$\nabla \times \vec{H} = 4\pi \sigma \vec{E} + 4\pi \left(4\pi \sigma \chi_e \vec{E}\right). \tag{118}$$

Substituting the polarization vector $\vec{P}$,

$$\vec{P} = \chi_e \vec{E}, \tag{119}$$

in Equation 118,

$$\nabla \times \vec{H} = 4\pi \sigma \vec{E} + 4\pi \left(4\pi \sigma \vec{P}\right) \tag{120}$$

$$= 4\pi \vec{J}_O + 4\pi \vec{J}_P, \tag{121}$$

where

$$\vec{J}_O = \sigma \vec{E} \tag{122}$$

is the Ohmic current flow, and

$$\vec{J}_P = 4\pi \sigma \vec{P} \tag{123}$$

is the non-Ohmic current flow. In the revised formulation, when current flows in a material due to an electric field $\vec{E}$, the polarization vector is also viewed as an "electric field", resulting in additional current flow.

Similar to how Equation 114 is written as Equation 115, the total volume current density, which is the sum of Equation 122 and Equation 123, can be written as

$$i = \frac{V}{R} + \frac{W}{R}, \tag{124}$$

which is the revised Ohm's law. In the above equation,

$$V = \int_A^B \vec{E} \cdot d\vec{l} \tag{125}$$

is the path integral of the electric field $\vec{E}$ along the segment of the wire from A to B, or the voltage between A and B, and

$$W = 4\pi \int_A^B \vec{P} \cdot d\vec{l} \tag{126}$$

is the path integral of the polarization vector $\vec{P}$ along the wire segment from A to B, which will be coined a new term, the polarization voltage. In the case where there is no polarization in the material where the current is flowing,

$$\epsilon_r = 1, \tag{127}$$

or equivalently,

$$\vec{P} = \vec{0}, \tag{128}$$

and Equation 124 reduces to the present Ohm's law,

$$i = \frac{V}{R}. \tag{129}$$

The "spurious eruption of $4\pi s$", in the words of Oliver Heaviside, can be removed by the process of rationalization, as explained in Reference [5].

XV. Revised Formulation in Electrostatics

In the case of electrostatics, where there are no time-varying sources and fields,

$$\vec{D}_F = \vec{0}, \tag{130}$$

since there is no time-varying $\vec{B}$ to generate a $\vec{D}_F$, as formulated in Faraday's law. Since there is no current flow, or magnetic materials in electrostatics,

$$\vec{B} = \vec{0} \tag{131}$$

$$\vec{H} = \vec{0}. \tag{132}$$

Equation 79 – Equation 89 reduce to

$$\nabla \cdot \vec{D}_C = 4\pi \rho \tag{133}$$

$$\nabla \times \vec{E}_C = \vec{0} \tag{134}$$

$$\vec{D}_C = \epsilon_r \vec{E}_C, \tag{135}$$

where the fields and sources are not a function of time. Writing the revised set of equations using $\vec{E}_C$, $\vec{E}_F$, $\vec{D}_C$, $\vec{D}_F$, results in the proper formulation of the conservative property of $\vec{E}_C$ in Equation 134, unlike Equation 37.

XVI. Revised Formulation in Magnetostatics

In the case of magnetostatics, where there are no time-varying sources or time-varying fields, no electric charges, and only a steady current flow,

$$\vec{D}_C = \vec{0} \tag{136}$$

$$\vec{D}_F = \vec{0} \tag{137}$$

$$\vec{E}_C = \vec{0} \tag{138}$$

$$\vec{E}_F = \vec{0}. \tag{139}$$

Equation 79 – Equation 89 reduce to

$$\nabla \cdot \vec{B} = 0 \tag{140}$$

$$\nabla \times \vec{H} = 4\pi \epsilon_r \vec{J}_O \tag{141}$$

$$\vec{B} = \mu_r \vec{H}. \tag{142}$$

The current continuity equation in Equation 67 reduces to

$$\nabla \cdot \left(\epsilon_r \vec{J}_O \right) = 0, \tag{143}$$

and the fields and the sources do not vary over time.

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