Nicolò Rigamonti

Proposal of solution of the Riemann hypothesis.

Abstract. In these papers we will try to face the Riemann hypothesis, basing on the disposition and the origin of couples of non-trivial zeros from the study of the functional equation of the Riemann zeta function.

0.0 Introduction

In this introduction we will concentrate upon the study of the zeros of functions expressed by a product of factors.
Let f(z) be a real or a complex function, expressed by a product of factors:

0.01 $\boldsymbol{f(z) = a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n}$, where z is a real or a complex number.

Now, we do the following reasoning.
For the zero-product property, we notice that:
$\mathrm{a}_1 = 0 \rightarrow \mathrm{f(z)} = 0$
$\mathrm{a}_2 = 0 \rightarrow \mathrm{f(z)} = 0$
$\mathrm{a}_3 = 0 \rightarrow \mathrm{f(z)} = 0$
↓
…………………………
↓
$\mathrm{a}_\mathrm{n} = 0 \rightarrow \mathrm{f(z)} = 0$
From this, we have that:
$\mathrm{f(z)} = 0$ if $\mathrm{a}_1 = 0 \vee \mathrm{a}_2 = 0 \vee \mathrm{a}_3 = 0 \vee \cdots \vee \mathrm{a}_\mathrm{n} = 0$
Thus, we deduce that the set S of the zeros of $\mathrm{f(z)}$ is equal to the union of the zeros of $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \ldots, \mathrm{a}_\mathrm{n}$.
Therefore, for these functions this property is valid:

" For a real or a complex function $\mathrm{f(z)}$ *with domain D, the set S of the zeros of the function* $\mathrm{f(z)}$ *is equal to the union between the sets of zeros of the single factors of the function* $\mathrm{f(z)}$*".* In formula:

0.02 $\boldsymbol{S = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n}$

From this property, we can divide the zeros of a function of the form **(0.01)** into groups, according to the factor, cancelled by the zero, in this way: $S_1 = set\ of\ the\ zeros\ of\ the\ first\ factor$ $S_2 = set\ of\ the\ zeros\ of\ the\ second\ factor$ $S_3 = set\ of\ the\ zeros\ of\ the\ third\ factor$
…………………………
$S_n = set\ of\ the\ zeros\ of\ the\ nth\ factor$

0 Zeta function: known properties. *[1]*

The Riemann zeta function is defined by the Dirichlet series:

0.1 $\boldsymbol{\zeta(s)= \sum_{n=1}^{\infty} \frac{1}{n^s}}$

for every complex number s with real part Re(s) greater than 1.

However, this function can be analytically continued to a holomorphic function on the whole complex s-plane, except for a simple pole at s=1, and it satisfies the following functional equation:

1 $\boldsymbol{\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)}$, where s is a complex number and Γ is the gamma function.

The zeta function has some zeros, called the trivial zeros, and others, known as the non-trivial zeros, which lie in the critical strip **0<Re(s)<1**, symmetrically about the real line and about the critical line.

Thus for every non-trivial zero s = $\sigma + it$:

0.2 $\boldsymbol{\exists!\, s_n \mid s_n = \sigma - it \wedge \exists!\, s_n \mid s_n = 1 - \sigma + it}$

The Riemann hypothesis is about the non-trivial zeros and it asserts that:

"Every non-trivial zero s has Re(s)=1/2".

In addition, for the zeta function **(0.1)** there is the following identity, which is the Euler product formula:

0.3 $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}}$,

where s is a complex number with real part Re(s) greater than 1 and p is a prime number.

Proof p(0) *[Euler]***:**

We consider the zeta function **(0.1)**

$\zeta(s)= \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$

We multiply it by the factor $\frac{1}{2^s}$

$\frac{1}{2^s}\zeta(s)= \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$

Subtracting the second equation from the first we get:

$(1 - \frac{1}{2^s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$

Doing this, we have removed all elements that have a factor of 2.

Repeating the same procedure for the next terms:

$(1 - \frac{1}{7^s})(1 - \frac{1}{5^s})\left(1 - \frac{1}{3^s}\right)(1 - \frac{1}{2^s})\,\zeta(s) = 1$

In this way, we are progressively removing all the multiples of every term after 1, which is a prime number since it is not a multiple of any lesser number. Therefore, the numbers of the product are all primes.

Repeating infinitely for $\frac{1}{p^s}$, where p is a prime, we obtain:

$\ldots (1 - \frac{1}{11^s})\,(1 - \frac{1}{7^s})(1 - \frac{1}{5^s})\left(1 - \frac{1}{3^s}\right)(1 - \frac{1}{2^s})\,\zeta(s) = 1$

Dividing both sides by everything but $\zeta(s)$,we get:

$\zeta(s) = \frac{1}{(1-\frac{1}{2^s})} \frac{1}{\left(1-\frac{1}{3^s}\right)} \frac{1}{(1-\frac{1}{5^s})} \frac{1}{(1-\frac{1}{7^s})} \frac{1}{(1-\frac{1}{11^s})} \cdots$,

which is equivalent to:

$\zeta(s) = \prod_{p\ prime} \frac{1}{1-p^{-s}}$ **(0.3)**

From the Euler product formula **(0.3)** we deduce an important consequence:

(0.4) *"the zeta function has no zeros in the half-plane Re(s)>1".*

1 Study of the functional equation

In order to study the functional equation, we factor it.

1.1 factor: 2^s

The factor 2^s is an exponential function of the form y=a^s, with a=2 and s ∈ ℂ. Since the base of the exponent is different from 0, there is no value of s such that cancels the factor 2^s and therefore the functional equation **(1)**. Thus:

1.11 $\nexists$ **s** $\in \boldsymbol{C} |\ \mathbf{2^S = 0} \ \rightarrow (\mathbf{1}) = \mathbf{0}$

1.2 factor: π^{s-1}

Doing a similar reasoning to that used for the factor **(1.1),** we can assert that there is no value of s such that cancels the factor π^{s-1} and therefore the functional equation **(1).** Thus:

1.21 $\nexists\ \boldsymbol{s}\ \in \boldsymbol{C} |\ \boldsymbol{\pi^{S-1} = 0} \ \rightarrow (\mathbf{1}) = \mathbf{0}$

1.3 factor: sin($\frac{\pi\ s}{2}$)

Since the factor $\sin\left(\frac{\pi s}{2}\right)$ is a goniometric function of the form y=sin(x), it cancels the functional equation **(1)** when the argument of the sin equals to kπ , with k$\in \mathbb{Z}$.

In this case we have that:

$\frac{\pi s}{2} = k\pi$

$\pi s = 2k\pi$

1.31 $\boldsymbol{s = 2k}$**, with** $\boldsymbol{k} \in \mathbb{Z}$

Because the zeta function **(0.1)** has no zeros for **(0.4)**, the formula **(1.31)** becomes:

1.32 s = -2n, with n $\in \mathbb{N} - \{\mathbf{0}\}$

Thus, when s = -2n, with n $\in \mathbb{N} - \{0\}$, the functional equation **(1)** equals 0. Therefore:

1.33 $\boldsymbol{\zeta(-2n) = 0}$**, with n** $\in \mathbb{N} - \{\mathbf{0}\}$

The trivial zeros of the zeta function derive from the formula **(1.33)**.

1.4 factor: $\boldsymbol{\Gamma(1-s)}$ *[2]*

Since the gamma function has the property to be non- zero everywhere, there is no value of s such that cancels the factor $\Gamma(1-s)$ and therefore the functional equation **(1)**. Thus:

1.41 $\nexists$ **s** $\in C|\ \boldsymbol{\Gamma(1-s)=0 \rightarrow (1)=0}$

1.5 factor: $\boldsymbol{\zeta(1-s)}$

[In the next paragraph we will concentrate upon the factor ***(1.5)****].*

2 Study of the factor $\boldsymbol{\zeta(1-s)}$

In order to study this factor, we will think about the following question:
"What happen when $\zeta(1-s)$=0, canceling the functional equation ***(1)****?"*
We formulate two hypothesis:

Hypothesis A: ***"If*** $\boldsymbol{\zeta(1-s)}$***=0, then*** $\boldsymbol{\zeta(1-s) \neq \zeta(s)}$**".**

Hypothesis B: ***"If*** $\boldsymbol{\zeta(1-s)}$***=0, then*** $\boldsymbol{\zeta(1-s) = \zeta(s)}$**".**

From the functional equation we demonstrate that the hypothesis **A** is false, while the hypothesis **B** is true.

Proof p(1):

Considered the functional equation **(1)**, if $\zeta(1-s)$*=0*, for the zero-product property, the functional equation equals 0. From this, we work out the following relation:

2.1 $\boldsymbol{\zeta(1-s)}$***=0*** $\rightarrow \boldsymbol{\zeta(s)}$ ***=0***

In fact:

If $\zeta(1-s)$*=0*

$\zeta(s) = 2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$

$\zeta(s) = 2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\ 0$

$\zeta(s)$ =0

Therefore, $\zeta(1-s)$*=0* $\rightarrow \zeta(s)$ *=0* **(2.1)**

From the relation **(2.1)**, the hypothesis **(A)** is false, whilst the hypothesis (**B**) is true:

(A) $\zeta(1-s)$*=0* $\rightarrow \zeta(s)$ *=0* $\rightarrow \zeta(1-s) \neq \zeta(s) \rightarrow (false\ for\ proof\ \boldsymbol{p}(\mathbf{1}))$.

(B) $\zeta(1-s)$*=0* $\rightarrow \zeta(s)$ *=0* $\rightarrow \zeta(1-s) = \zeta(s) \rightarrow (true\ for\ proof\ \boldsymbol{p}(\mathbf{1}))$.

From the veracity of the hypothesis **(B)** we deduce a key condition satisfied when $\zeta(1-s)$*=0* that is:

2.2 $\boldsymbol{\zeta(s) = \zeta(1-s)}$

From the condition **(2.2)** we deduce an important consequence:
"The trivial zeros of the zeta function don't derive from the factor $\zeta(1-s)$".

Proof p(2):

Let s = -2n, with n $\in \mathbb{N}-\{0\}$, we substitute it in the functional equation **(1)**:

$$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin\left(\frac{\pi(-2n)}{2}\right)\Gamma\big(1-(-2n)\big)\zeta\big(1-(-2n)\big)$$

$$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin(-n\pi)\,\Gamma(1+2n)\zeta(1+2n)$$

Done the substitution and the related calculations, $\zeta(s)$ and $\zeta(1-s)$ have become respectively $\zeta(-2n)$ and $\zeta(1+2n)$.

For the formula **(1.33)** $\zeta(-2n) = 0$, while $\zeta(1+2n) \neq 0$, because the zeta function of the form **(0.1)** has no zeros.

Therefore $\zeta(-2n) \neq \zeta(1+2n)$ and thus $\zeta(s) \neq \zeta(1-s)$.

Consequently the condition **(2.2)** is not satisfied and so the trivial zeros don't derive from the factor **(1.5)**, but only from **(1.3).**

3 Study of the condition $\zeta(s) = \zeta(1-s)$ in the critical strip

We consider the functional equation **(1).** From what we have said in the previous paragraphs, we deduce that the non-trivial zeros of the zeta function don't derive from the factors **(1.1), (1.2), (1.4)**, because they don't cancel the functional equation, and even from the factor **(1.3)**, from which the sole trivial zeros derive. Thus they only derive from the factor **(1.5)** and satisfy the condition **(2.2)**.
Since the functional equation **(1)** of the zeta function is of the form **(0.01)**, we divide its zeros in groups:

$S_1 = \emptyset$

$S_2 = \emptyset$

$S_3 = set\ of\ trivial\ zeros$

$S_4 = \emptyset$

$S_5 = set\ of\ non\text{-}trivial\ zeros$

And for the property **(0.02)** we get:

$S = S_3 \cup S_5$, which is a coherent result.

Now, we suppose that s_0 is a non-trivial zero with real part $\sigma \neq \frac{1}{2}$ and imaginary part it:

$s_0 = \sigma + it$

We substitute s_0 in the condition **(2.2)**:

$\zeta(s_0) = \zeta(1-s_0)$

From this, we obtain a second non-trivial zero s_1, which is equal to 1-s_0:

$s_1 = 1 - s_0 = (1-\sigma) - it$

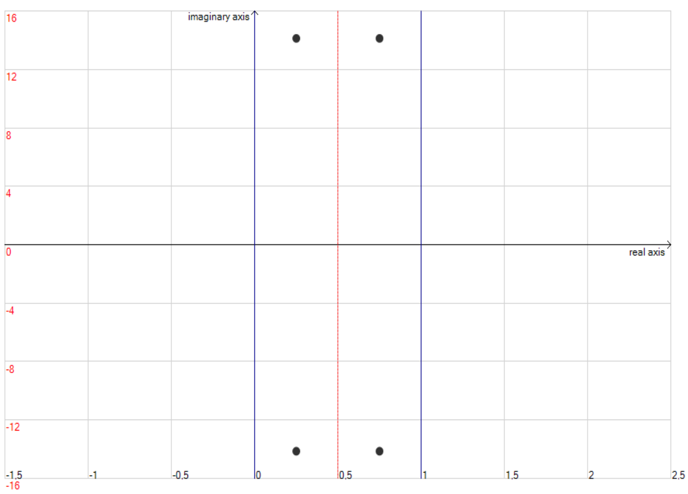
From the property **(0.2)** of the non-trivial zero we obtain s_2 and s_3:

$s_2 = \sigma - it$

$s_3 = (1-\sigma) + it$

We represent all the non-trivial zeros in the complex plane:

Picture **1**



We have said that the functional equation of the zeta function is of the form **(0.01)**,however, it represents a particular case. In fact the zeta function is both the considered function and the factor **(1.5).**

For this, we will consider the zeros, which cancels the factor **(1.5)**, not as single zeros, such as the trivial zeros, for which s_0=-2, s_1=-4, s_2=-6,..., but as single couples of zeros (in this case, one zero is from $\zeta(s)$ and the other from $\zeta(1-s)$):

$(s_{0(1)}; s_{0(2)})$, $(s_{1(1)}; s_{1(2)})$,($s_{2(1)}; s_{2(2)}$),....

So, for the relation **(2.2)** we deduce that all the zeros ∈ S_5, **with imaginary part** $\pm$ ***it*** **and equal *t***, have the property to be symmetric about the point P($\frac{1}{2}$; 0). In fact for them the following relation is true:

3.2 $\begin{cases} \boldsymbol{\zeta(s_k) = \zeta(s)} \\ \boldsymbol{\zeta(s_j) = \zeta(1-s)} \end{cases}$

We have shown that if s_0 is a non-trivial zero with real part $\sigma \neq \frac{1}{2}$, there will be another three non-trivial zeros (s_1, s_2, s_3) such that we will have the situation represented in the picture **1.**

Now, we consider all the possible couples of non-trivial zeros:

$\{(s_0; s_1), (s_0; s_2), (s_0; s_3), (s_1; s_2), (s_1; s_3), (s_2; s_3)\}$

From the relation **(3.2)** we deduce that only the couples $(s_0; s_1)$ and $(s_2; s_3)$ satisfy it.

Therefore the disposition of the non-trivial zeros, shown in the picture **1**, is an absurd because the couples $(s_0; s_2)$, $(s_0; s_3)$, $(s_1; s_2)$ and $(s_1; s_3)$ don't satisfy the relation **(3.2)**; in fact, since they aren't symmetric about the point P($\frac{1}{2}$; 0), but about the real line, for $(s_0; s_2)$ and $(s_1; s_3)$, and about the critical line x=$\frac{1}{2}$ for $(s_0; s_3)$ and $(s_1; s_2)$, they $\notin$ S_5.

The disposition of the zeros, shown in the picture **1**, would be true if the functional equation **(0.1)** was of the form:

$\zeta(s) = 2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\zeta(\bar{s})\zeta(1-\bar{s})$

In this case we would have two new sets of zeros: S_6 and S_7, obtaining:

$S_1 = \emptyset$

$S_2 = \emptyset$

$S_3 = set\ of\ trivial\ zeros$

$S_4 = \emptyset$

$S_5 = set\ of\ non\text{-}trivial\ zeros$ $[couples\ of\ zeros\ , simmetric\ about\ the\ point\ \mathrm{P}\left(\frac{1}{2}; 0\right)]$

$S_6 = set\ of\ couples\ of\ zeros, simmetric\ about\ the\ real\ line. Here\ the\ couples (s_0; s_2)\ and\ (s_1; s_3), named\ before, would\ be\ included.$

$S_7 = set\ of\ couples\ of\ zeros, simmetric\ about\ the\ critical\ line. Here\ the\ couples (s_0; s_3)\ and\ (s_1; s_2), named\ before, would\ be\ included.$

However, because the zeta function satisfy the functional equation **(0.1)**, there aren't couples of zeros symmetric about the real line or about the critical line, but only the couples of zeros ∈ S_5, which, for what we have said previously, must be symmetric about the point P($\frac{1}{2}$; 0).

The only case, for which the disposition of the zeros in the picture **1** is true, is when s_0 and s_1 coincide respectively with s_3 and s_2.

Thus we have:

$s_0 = s_3$

$\sigma + it = (1 - \sigma) + it$

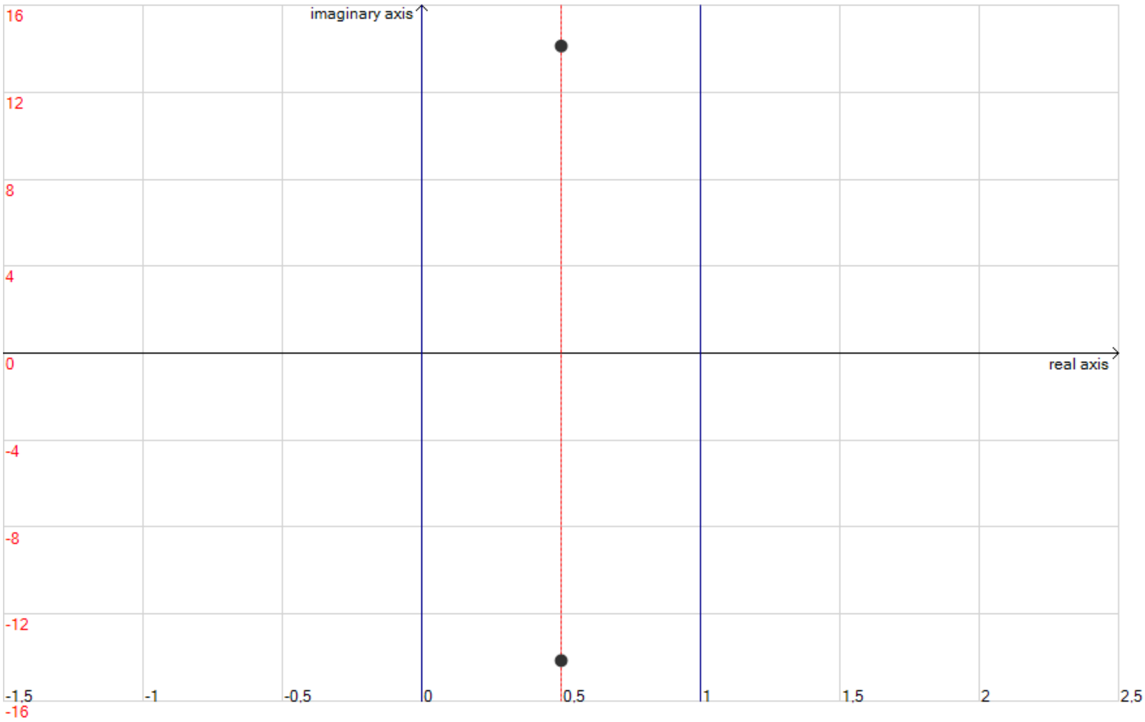
$\sigma = 1 - \sigma$

$2\,\sigma = 1$

$\sigma = \frac{1}{2}$

We represent the situation in the complex plane:

Picture **2**



From this calculus we have obtained that $s_0 = s_3$ when $\sigma = \frac{1}{2}$ and this confirms the veracity of the Riemann hypothesis. In fact the Riemann hypothesis is the only case such that there will be only couples of non-trivial zeros symmetric about the point P($\frac{1}{2}$; 0). Actually, the non-trivial zeros of the zeta function are symmetric about the real line as represented in the picture **2**; however they are also symmetric about the point P($\frac{1}{2}$; 0) at the same time and this is the most important thing.

4 Conclusions

In conclusion, only at this point , we can substitute a complex number of the form $\frac{1}{2} + it$ in the condition **(2.2)**, getting:

3.3 $\boldsymbol{\zeta\left(\frac{1}{2} + it\right) = \zeta\left(\frac{1}{2} - it\right)}$

and saying that:

3.4 $\boldsymbol{\zeta\left(\frac{1}{2} \pm it\right) = 0}$ for all the non-trivial zeros.

References

[1] Wikipedia, Riemann zeta function: https://en.wikipedia.org/wiki/Riemann_zeta_function

[2] Wikipedia, Gamma function: https://en.wikipedia.org/wiki/Gamma_function