The Complexity of Robust and Resilient $k$-Partition Problems

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Abstract

In this paper, we study a $k$-partition problem where a set of agents must be partitioned into a fixed number of $k$ non-empty coalitions. The value of a partition is the sum of the pairwise synergies inside its coalitions. Firstly, we aim at computing a partition that is robust to failures from any set of agents with bounded size. Secondly, we focus on resiliency: when a set of agents fail, others can be moved to replace them.

We introduce relevant notation for $i$-partition. The robustness value of $\pi$, $\sum_{i,j} w(i,j)$ satisfying $w(i,j) = w(j,i)$ and $w(i,i) = 0$ for any $i, j \in N$. The value that an agent $i$ gives to a $k$-partition $\pi$ is defined by $v_i(\pi) = \sum_{j \in \pi(i)} w(i,j)$, the sum of synergies between her and the other agents in the same coalition. It follows that the utilitarian value $v(\pi) = \frac{1}{2} \sum_{i,j} v_i(\pi)$ can also be defined as $v(\pi) = \sum_{C \subseteq \pi} \sum_{\{i,j\} \subseteq C} w(i,j)$, the sum of synergies between pairs in a same coalition. If some coalition is empty, then $\pi$ is not a $k$-partition, but a partition: in that case, we extend $v$ to $v(\pi) = -\infty$.

Any set $M \subseteq N$ of at most $m \in \mathbb{N}_{>0}$ agents might fail. Given a $k$-partition $\pi$ of set $N$, resulting partition $\pi_{-M}$ of set $N \setminus M$ is defined as $\{C \setminus M \mid C \in \pi\}$ (and might contain empty coalitions). The robustness value of $\pi$ is

$$v^-(\pi) = \min_{M \subseteq N, |M| \leq m} \{v(\pi_{-M})\}.$$

To obtain $v^-(\pi) \neq -\infty$, every coalition in $\pi$ shall contain at least $m+1$ agents, so that no coalition of $\pi_{-M}$ is empty.

Once agents $M \subseteq N$ fail and $N \setminus M$ remain, replacements are possible, by moving any subset $R \subseteq N \setminus M$ of other agents, with a replacement function $\rho : R \to \pi$. For agent $i \in R$, $\rho(i)$ is her new coalition. For coalition $C \in \pi$, $\rho(C)$ are the agents newly moved to $C$. Hence, given $k$-partition $\pi = \{C_1, \ldots, C_k\}$, set $M \subseteq N$ of $n$ failing agents and replacements $R \subseteq N \setminus M$, $|R| \leq m$ and $\rho : R \to \pi$, the repaired partition $\pi_{-M}^R = \{(C \cup \rho(C)) \setminus (M \cup R) \mid C \in \pi\}$. Given $k$-partition $\pi$, its resiliency value is

$$v^+ (\pi) = \min_{M \subseteq N, |R| \leq m} \left\{v(\pi_{-M}^R)\right\},$$

where $M \subseteq N$ has size $|M| \leq m$, $R \subseteq N \setminus M$ has size $|R| \leq m$, and $\rho : R \to \pi$ is a map. As long as $2m \leq n$, given a $k$-partition $\pi$ and a failing set $M$, there always exists a replacement $R, \rho$ that repairs partition $\pi_{-M}$ into a $k$-partition $\pi_{-M}^R$; hence, one always has $v^+(\pi) \neq -\infty$.

In such a simultaneous replacement, all agents in $M$ fail at once, after which all agents in $R$ are moved by $\rho$. We shall only require from $R, \rho$ that for every coalition $C \in \pi$, $(C \cup \rho(C)) \setminus (M \cup R)$ is non-empty. In a sequential replacement, there are up to $m$ rounds. The idea is that in each round $t \in [m]$, one new agent $f_t$ fails, and then one remaining agent $r_t$ is moved to coalition $\rho_t(r_t)$. Let $M_t = \{f_1, \ldots, f_t\}$ denote the agents that failed up to round $t$. It satisfies $f_t \in N \setminus M_{t-1}$ and $|M_t| = t$. Let $R_t = \{\ldots, r_t\}$ and $\rho_t : R_t \setminus M_t \to \pi$ denote replacements up to round $t$. While $r_t \in N \setminus M_t$ always holds, a same agent can be moved in two rounds, and an agent who moved can fail later, so that only $|R_t \setminus M_t| \leq t$ is true, instead of $|R_t \setminus M_t| = t$. Crucially, on any round $t \in [m]$, $\pi_{-M_t}^{R_t}$ has no empty coalition. In other words, if failure $f_t$ empties a coalition, then it shall be replaced by $r_t, \rho_t(r_t)$. A valid strategy $\sigma$ maps any history $(M_t, R_{t-1}, \rho_{t-1})$ (failures set $M_t$ and replacements $R_{t-1}, \rho_{t-1}$) to next replacement $(r_t, \rho_t(r_t)) = \sigma(M_t, R_{t-1}, \rho_{t-1})$, while never letting any coalition of $\pi_{-M_t}^{R_t}$ empty, for any round $t$. Strategy $\sigma$, for any sequence of failures $f_1, \ldots, f_m$ alternated with the replacements of $\sigma$, induces final replacements $R_m, \rho_m$.

We assume that the following concepts are common knowledge: decision problem, length function, polynomial-
time many-to-one reduction, hardness, completeness and classes P, NP, coNP, $\Sigma_2^p$, $\Pi_2^p$, $\Sigma_3^p$, PH and PSPACE.

**Definition 1.** We study this sequence of decision problems:

- **ROBUST-$k$-PART/VERIFY**
  Given $n$ agents, synergies $w$, a number $m$ of failing agents, a $k$-partition $\pi$ and a threshold $\theta \in \mathbb{Z}$, does robustness value $v^-(\pi)$ satisfy $v^-(\pi) \geq \theta$?

- **ROBUST-$k$-PART**
  Given $n$ agents, synergies $w$, a number $m$ of failing agents and a threshold $\theta \in \mathbb{Z}$, is there a $k$-partition $\pi$ with robustness value satisfying $v^-(\pi) \geq \theta$?

- **SimRes-$k$-PART/VERIFY**
  Given $n$ agents, synergies $w$, $k$-partition $\pi$, set $M \subseteq N$ of $m$ failures and threshold $\theta \in \mathbb{Z}$, is there a replacement $R \subseteq N \setminus M$, $|R| \leq m$, $\rho : R \rightarrow \pi$ such that $v(\pi^{R,\rho}) \geq \theta$?

- **SimRes-$k$-PART**
  Given $n$ agents, synergies $w$, $k$-partition $\pi$ and threshold $\theta \in \mathbb{Z}$, does resiliency value $v^+(\pi)$ satisfy $v^+(\pi) \geq \theta$?

- **SeqRes-$k$-PART/STR**
  Given $n$ agents, synergies $w$, $k$-partition $\pi$ and threshold $\theta \in \mathbb{Z}$, does a valid strategy $\sigma$ exist such that for any sequence of failures $f_1, \ldots, f_m$ the final replacements $R_m, \rho_m$ induced by $\sigma$ are such that $v(\pi^{R_m,\rho_m}) \geq \theta$?

- **SeqRes-$k$-PART**
  Given $n$ agents, synergies $w$, $k$-partition $\pi$ and threshold $\theta \in \mathbb{Z}$, does a $k$-partition $\pi$ and valid strategy $\sigma$ exist such that for any sequence of failures $f_1, \ldots, f_m$, the final replacements $R_m, \rho_m$ induced by $\sigma$ are such that $v(\pi^{R_m,\rho_m}) \geq \theta$?

### The Complexity of Robust $k$-Partition

In this section, we settle the computational complexity of robust $k$-partition as complete for the second level of the polynomial hierarchy.

**Theorem 1.** **ROBUST-$k$-PART/VERIFY** is coNP-complete. (It holds even for $k = 1$, synergies $w$ in $\{0, 1\}$ and $\theta = 1$.)

**Proof.** Decision problem **ROBUST-$k$-PART/VERIFY**, given $n$ agents, synergies $w$, a number $m$ of failing agents, a $k$-partition $\pi$ and a threshold $\theta \in \mathbb{Z}$, asks whether:

$$\forall M \subseteq N, |M| \leq m, \quad v(\pi_{\lnot M}) \geq \theta.$$  

This problem is in class coNP, since for any no-instance, a failing set $M$ such that $v(\pi_{\lnot M}) \leq \theta$ is a no-certificate verifiable in polynomial-time. We show coNP-hardness by complementary reduction from MinVertexCover. Let graph $G = (V, E)$ and threshold $m \in \mathbb{N}$ be any instance of MinVertexCover, which asks whether there exists a subset $U \subseteq V, |U| \leq m$ such that $\forall \{i, j\} \in E, i \in U$ or $j \in U$, i.e. every edge is covered by a vertex in $U$. We reduce it to a ROBUST-$k$-PART/VERIFY instance with agents $N \equiv V$, synergies $w(i, j) \in \{0, 1\}$ equal to one if and only if $\{i, j\} \in E$ (otherwise zero) and threshold $\theta = 1$. Our $k$-partition $\pi$ is the grand coalition ($k = 1$).

![Figure 1: We reduce any instance of MAXMINVERTEXCOVER $G = (V, E)$ where $V = \bigcup_{i \in I} \{V_{i, 0} \cup V_{i, 1}\}$ and $m$ is a threshold, to the following instance of ROBUST-$k$-PART.](image)

Figure 1: We reduce any instance of MAXMINVERTEXCOVER $G = (V, E)$ where $V = \bigcup_{i \in I} \{V_{i, 0} \cup V_{i, 1}\}$ and $m$ is a threshold, to the following instance of ROBUST-$k$-PART. Agents $N \equiv V$ are identified with vertices, hence can be partitioned the same into $N = \bigcup_{i \in I} (N_{i, 0} \cup N_{i, 1})$. We fix $k = 2$ coalitions and choose a large number $L$, e.g. $L = n^2$. For every $\{i, j\} \in P_2(N)$, if $\{i, j\} \in E$, we define synergy $w(i, j) = 2$; otherwise if $\{i, j\} \notin E$, we define $w(i, j) = 1$; but for every $\ell \in I$ and every $\{i, j\} \in N_{i, 0} \times N_{\ell, 1}$, where we define synergy $w(i, j) = -L$. Up to $2m$ agents might fail, and the threshold is defined in the proof.

(yes$\Rightarrow$no) If there exists a vertex cover $U \subseteq V, |U| \leq m$, then failing set $M = U$ is such that any synergy $w(i, j)$ equal to one has $i \in M$ or $j \in M$, hence disappears from $\pi$ in $\pi_{\lnot M}$, and value is $v(\pi_{\lnot M}) \leq 0$.

(yes$\Rightarrow$no) If there is a failing set $M \subseteq N, |M| \leq m$ such that $v(\pi_{\lnot M}) \leq 0$, then any synergy $w(i, j)$ equal to one has $i \in M$ or $j \in M$. Therefore, $U \equiv M$ is a vertex cover.

If $k > 1$, this result still holds (when coalitions are larger than any failing set). It suffices to copy the construction above into $k$ identical coalitions and ask for at most $km$ failing agents. The yes$\Rightarrow$no part is trivially the same. Concerning yes$\Leftarrow$no, even though a failing set of size at most $km$ might be unequivocally distributed between the $k$ identical coalitions, the coalition containing the smallest failing subset gives a vertex cover smaller than $m$.

**Theorem 2.** **ROBUST-$k$-PART** is $\Sigma_2^p$-complete. (It holds even for $k = 2$ coalitions and $w \in \{-n^2, 1, 2\}$)

**Proof.** Decision problem **ROBUST-$k$-PART**, given $n$ agents, a number $k$ of coalitions, synergies $w$, a number $m$ of failures and a threshold $\theta \in \mathbb{Z}$, asks whether:

$$\exists k$-partition $\pi, \quad \forall M \subseteq N, |M| \leq m, \quad v(\pi_{\lnot M}) \geq \theta.$$  

It lies in class $\Sigma_2^p$, since for yes-instances, such a $k$-partition $\pi$ is a certificate that can be verified by an NP-oracle on remaining coNP problem **ROBUST-$k$-PART/VERIFY**. We show $\Sigma_2^p$-hardness by a complementary reduction from $\Pi_2^p$-complete problem MAXMINVERTEXCOVER, defined as follows. Given a finite graph $G = (V, E)$ where vertices are partitioned by a finite index set $I$ into $V = \bigcup_{i \in I} \{V_{i, 0} \cup V_{i, 1}\}$, for a function $p : I \rightarrow \{0, 1\}$, we define $V^{(p)} = \bigcup_{i \in I} V_{i, p(i)}$ and $G^{(p)} = G[V^{(p)}]$. Given a threshold
Artificial Intelligence

Forming k coalitions and facilitating relationships in social


Kraus, S., and Yokoo, M. 2017. private communication.


Related Work

Partitioning of a set into (non-empty) subsets may also be referred as coalition structure formation of a set of agents into coalitions. When a number of coalitions k is required and there are synergies between vertices/agents, this problem is referred as k-cut, or k-way partition, where one minimizes the weight of edges/synergies between the coalitions, or maximizes it inside the coalitions. For positive weights and k ≥ 3, this problem is NP-complete (Dahlhaus et al. 1992), when one vertex is fixed in each coalition. For positive weights and fixed k, a polynomial-time O(nk2T(n, m)) algorithm exists (Goldschmidt and Hochbaum 1994), when no vertex is fixed in coalitions, and where T(n, m) is the time to find a minimum (s, t) cut on a graph with n vertices and m edges. When not too many negative synergies exist (that is, negative edges can be covered by O(log(n)) vertices), an optimal k-partition can be computed in polynomial-time (Sless et al. 2018). Various formulations of the robust and resilient problem studied were initially proposed by (Kraus and Yokoo 2017). The complexity results in this paper result from an original work in July and August 2018, between Anisse Ismaili and Eimi Watanabe.

The Complexity of Resilient k-Partition

We conjecture Theorems 3-7:

Theorem 3. SIMRES-k-PART/VERIF2 is NP-complete.
Theorem 4. SIMRES-k-PART/VERIF is ΠP 2-complete.
Theorem 5. SIMRES-k-PART is ΣP 3-complete.
Theorem 6. SEQRES-k-PART/STR is PSPACE-complete.
Theorem 7. SEQRES-k-PART is PSPACE-complete.