Infinite Product Representations for Some Surd Numbers and Infinite Sum Representations for Some Logarithm Constants

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. I derive identities for some surd numbers, involving gamma functions; thence, I have represented them as infinite products.

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1. Introduction

In present paper, I derive the identity below

$$n^{\ell-1}\cdot\sqrt[n]{n}=\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\prod_{k=1}^{n-1}\frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)},$$

which enabled me to prove the following infinite products

$$2^{\ell-1/2}=\prod_{k=0}^{\infty}\left(\frac{4k+2\ell+1}{4k+2}\right)\left(\frac{4k+2\ell+3}{4k+4\ell+2}\right),$$

$$3^{\ell-2/3}=\prod_{k=0}^{\infty}\left(\frac{9k+3\ell+1}{9k+3}\right)\left(\frac{9k+3\ell+4}{9k+6}\right)\left(\frac{9k+3\ell+7}{9k+9\ell+3}\right),$$

$$4^{\ell-3/4}=2^{2\ell-3/2}=$$
$$=\prod_{k=0}^{\infty}\left(\frac{16k+4\ell+1}{16k+4}\right)\left(\frac{16k+4\ell+5}{16k+8}\right)\left(\frac{16k+4\ell+9}{16k+12}\right)\left(\frac{16k+4\ell+13}{16k+16\ell+4}\right),$$

$$5^{\ell-4/5}=\prod_{k=0}^{\infty}\left(\frac{25k+5\ell+1}{25k+25\ell+5}\right)\left(\frac{25k+5\ell+6}{25k+5}\right)\left(\frac{25k+5\ell+11}{25k+10}\right)$$
$$\left(\frac{25k+5\ell+16}{25k+15}\right)\left(\frac{25k+5\ell+21}{25k+20}\right),$$

and so on; more specifically, I get

$$\sqrt{2}=\prod_{k=0}^{\infty}\left(\frac{4k+3}{4k+2}\right)\left(\frac{4k+5}{4k+6}\right)$$
$$=\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{9}{10}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{15}{14}\cdot\frac{17}{18}\cdot\ldots,$$

$$\sqrt[3]{3}=\prod_{k=0}^{\infty}\left(\frac{9k+4}{9k+3}\right)\left(\frac{9k+7}{9k+6}\right)\left(\frac{9k+10}{9k+12}\right)$$
$$=\frac{4}{3}\cdot\frac{7}{6}\cdot\frac{10}{12}\cdot\frac{13}{12}\cdot\frac{16}{15}\cdot\frac{19}{21}\cdot\frac{22}{21}\cdot\frac{25}{24}\cdot\frac{28}{30}\cdot\ldots,$$

$$\sqrt[4]{4}=\sqrt{2}=\prod_{k=0}^{\infty}\left(\frac{16k+5}{16k+4}\right)\left(\frac{16k+9}{16k+8}\right)\left(\frac{16k+13}{16k+12}\right)\left(\frac{16k+17}{16k+20}\right)$$
$$=\frac{5}{4}\cdot\frac{9}{8}\cdot\frac{13}{12}\cdot\frac{17}{20}\cdot\frac{21}{20}\cdot\frac{25}{24}\cdot\frac{29}{28}\cdot\frac{33}{36}\cdot\ldots,$$

$$\sqrt[5]{5}=\prod_{k=0}^{\infty}\left(\frac{25k+6}{25k+5}\right)\left(\frac{25k+11}{25k+10}\right)\left(\frac{25k+16}{25k+15}\right)\left(\frac{25k+21}{25k+20}\right)\left(\frac{25k+26}{25k+30}\right)$$
$$=\frac{6}{5}\cdot\frac{11}{10}\cdot\frac{16}{15}\cdot\frac{21}{20}\cdot\frac{26}{30}\cdot\frac{31}{30}\cdot\frac{36}{35}\cdot\frac{41}{40}\cdot\frac{46}{45}\cdot\frac{51}{55}\cdot\ldots,$$

and so on.

On the other hand, I derived the following infinite sum formulas for some logarithm constants

$$\frac{\ln 2}{2}=\sum_{k=0}^{\infty}\frac{4\ell^2+(8k+4)\ell+1}{(2k+2\ell+1)(4k+2\ell+1)(4k+2\ell+3)}$$

and

$$\frac{\ln 3}{3}=\sum_{k=0}^{\infty}\frac{81k^2(6\ell-1)+18k(18\ell^2+21\ell-1)+(6\ell+11)(3\ell+1)^2}{(3k+3\ell+1)(9k+3\ell+1)(9k+3\ell+4)(9k+3\ell+7)}.$$

more specifically, I get

$$\frac{\ln 2}{2}=\sum_{k=0}^{\infty}\frac{8k+9}{(2k+3)(4k+3)(4k+5)}$$
$$=\frac{9}{3\cdot 3\cdot 5}+\frac{17}{5\cdot 7\cdot 9}+\frac{25}{7\cdot 11\cdot 13}+\ldots$$

and

$$\frac{\ln 3}{3}=\sum_{k=0}^{\infty}\frac{405k^2+684k+272}{(3k+4)(9k+4)(9k+7)(9k+10)}$$
$$=\frac{272}{4\cdot 4\cdot 7\cdot 10}+\frac{1361}{7\cdot 13\cdot 16\cdot 19}+\frac{3260}{10\cdot 22\cdot 25\cdot 28}+\ldots$$

2. PRELIMINARIES

I use the following classical formula, [1, Section 12.13; 2], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

Corollary 2.1. *If k is a positive integer and $a_1+a_2+\ldots+a_k=b_1+b_2+\ldots+b_k$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty}\frac{(\ell+a_1)\cdot\ldots\cdot(\ell+a_k)}{(\ell+b_1)\cdot\ldots\cdot(\ell+b_k)}=\frac{\Gamma(b_1)\cdot\ldots\cdot\Gamma(b_k)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_k)}. \tag{2.1}$$

Proof. See [1, Section 12.13]. □

The Gauss multiplication formula for gamma function assures me that

Theorem 2.2. *(Gauss)*

$$\forall z\notin\left\{-\frac{m}{n}: m\in\mathbb{N}\right\}: \prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)=(2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz), \tag{2.2}$$

where $\Gamma(z)$ denotes the gamma function.

Proof. See [3]. □

3. The Main Theorem

3.1. The numbers of form $n^{\ell-1}\cdot\sqrt[n]{n}$ like the finite product of gamma functions.

Theorem 3.1. *If $\ell\in\mathbb{N}$ and $n\in\mathbb{N}_{\geqslant 2}$, then*

$$n^{\ell-1}\cdot\sqrt[n]{n}=\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\prod_{k=1}^{n-1}\frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)}. \tag{3.1}$$

Proof. From Theorem 2.2, I obtain with a bit of manipulation

$$\begin{aligned}\Gamma(z)\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right)&=(2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz)\\ \Rightarrow\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right)&=(2\pi)^{(n-1)/2}n^{1/2-nz}\frac{\Gamma(nz)}{\Gamma(z)}.\end{aligned} \tag{3.2}$$

Replace z by $\left(\frac{\ell}{n}+\frac{1}{n^2}\right)$ in (3.2)

$$\begin{aligned}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)&=(2\pi)^{(n-1)/2}n^{1/2-\ell-1/n}\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\\ \Rightarrow\frac{1}{\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)}&=(2\pi)^{-(n-1)/2}n^{-1/2+\ell+1/n}\frac{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}{\Gamma\left(\ell+\frac{1}{n}\right)}.\end{aligned} \tag{3.3}$$

From right hand side of the Thoerem 3.1 and the right hand side of (3.3), it follows that

$$\begin{aligned}\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\prod_{k=1}^{n-1}\frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)}&=\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\cdot\frac{\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)}{\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)}\\&=\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\cdot\left[(2\pi)^{-(n-1)/2}n^{-1/2+\ell+1/n}\frac{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}{\Gamma\left(\ell+\frac{1}{n}\right)}\right]\cdot\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\\&=(2\pi)^{-(n-1)/2}n^{-1/2+\ell+1/n}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right).\end{aligned}\tag{3.4}$$

On the other hand, replace z by $1/\ell$, let ℓ tends to infinity in both members of (3.2)

$$\lim_{\ell\to\infty}\prod_{k=1}^{n-1}\Gamma\left(\frac{1}{\ell}+\frac{k}{n}\right)=\lim_{\ell\to\infty}(2\pi)^{(n-1)/2}n^{1/2-n/\ell}\frac{\Gamma\left(\frac{n}{\ell}\right)}{\Gamma\left(\frac{1}{\ell}\right)}.\tag{3.5}$$

Note that

$$\lim_{\ell\to\infty}\Gamma\left(\frac{1}{\ell}+\frac{k}{n}\right)=\Gamma\left(\frac{k}{n}\right),\tag{3.6}$$

$$\lim_{\ell\to\infty}n^{-n/\ell}=1\tag{3.7}$$

and

$$\lim_{\ell\to\infty}\frac{\Gamma\left(\frac{n}{\ell}\right)}{\Gamma\left(\frac{1}{\ell}\right)}=\frac{1}{n}.\tag{3.8}$$

From (3.5), (3.6), (3.7) and (3.8), I conclude that

$$\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)=\frac{(2\pi)^{(n-1)/2}n^{1/2}}{n}.\tag{3.9}$$

Now, from (3.4) and (3.9), it follows that

$$\begin{aligned}\frac{\Gamma\left(\ell+\frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n}+\frac{1}{n^2}\right)}\prod_{k=1}^{n-1}\frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n}+\frac{\ell}{n}+\frac{1}{n^2}\right)}&=(2\pi)^{-(n-1)/2}n^{-1/2+\ell+1/n}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\\&=(2\pi)^{-(n-1)/2}n^{-1/2+\ell+1/n}\left[\frac{(2\pi)^{(n-1)/2}n^{1/2}}{n}\right]\\&=\frac{n^{\ell+1/n}}{n}=n^{\ell-1}\cdot\sqrt[n]{n},\end{aligned}$$

which is the desired result of the left hand side of the Theorem above. This completes the proof. □

4. Some surd Numbers and Infinite Product Representation

4.1. The $\sqrt{2}$ and some multiples of the form $2^{\ell-1/2}$.

Corollary 4.1. *If* $\ell \in \mathbb{Z}^+$, *then*

$$2^{\ell-\frac{1}{2}} = \prod_{k=0}^{\infty} \left(\frac{4k+2\ell+1}{4k+2}\right)\left(\frac{4k+2\ell+3}{4k+4\ell+2}\right). \tag{4.1}$$

Proof. Set $n=2$ in Theorem 3.1 and find

$$\begin{aligned} 2^{\ell-1}\cdot\sqrt{2} &= \frac{\Gamma\left(\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}+\frac{1}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\ell}{2}+\frac{1}{4}\right)} \\ \Rightarrow 2^{\ell-1/2} &= \frac{\Gamma\left(\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}+\frac{1}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}+\frac{3}{4}\right)}. \end{aligned} \tag{4.2}$$

Note that

$$\ell+\frac{1}{2}+\frac{1}{2} = \frac{\ell}{2}+\frac{1}{4}+\frac{\ell}{2}+\frac{3}{4},$$

satisfy the condition $a_1+a_2=b_1+b_2$; $k=2$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and (4.2), it follows that

$$\begin{aligned} 2^{\ell-1/2} &= \prod_{k=0}^{\infty}\left(\frac{k+\frac{\ell}{2}+\frac{1}{4}}{k+\ell+\frac{1}{2}}\right)\left(\frac{k+\frac{\ell}{2}+\frac{3}{4}}{k+\frac{1}{2}}\right) \\ &= \prod_{k=0}^{\infty}\left(\frac{4k+2\ell+1}{4k+2}\right)\left(\frac{4k+2\ell+3}{4k+4\ell+2}\right), \end{aligned}$$

which is the desired result. □

Example 4.2. Let $\ell=1$ in Corollary 4.1 and encounter

$$\begin{aligned} \sqrt{2} &= \prod_{k=0}^{\infty}\left(\frac{4k+3}{4k+2}\right)\left(\frac{4k+5}{4k+6}\right) \\ &= \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{9}{10}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{15}{14}\cdot\frac{17}{18}\cdot\ldots \end{aligned}$$

Example 4.3. Let $\ell=2$ in Corollary 4.1 and encounter

$$\begin{aligned} 2\sqrt{2} &= \prod_{k=0}^{\infty}\left(\frac{4k+5}{4k+2}\right)\left(\frac{4k+7}{4k+10}\right) \\ &= \frac{5}{2}\cdot\frac{7}{10}\cdot\frac{9}{6}\cdot\frac{11}{14}\cdot\frac{13}{10}\cdot\frac{15}{18}\cdot\frac{17}{14}\cdot\frac{19}{22}\cdot\ldots \end{aligned}$$

Example 4.4. Let $\ell=3$ in Corollary 4.1 and encounter

$$\begin{aligned} 4\sqrt{2} &= \prod_{k=0}^{\infty}\left(\frac{4k+7}{4k+2}\right)\left(\frac{4k+9}{4k+14}\right) \\ &= \frac{7}{2}\cdot\frac{9}{14}\cdot\frac{11}{6}\cdot\frac{13}{18}\cdot\frac{15}{10}\cdot\frac{17}{22}\cdot\frac{19}{14}\cdot\frac{21}{26}\cdot\ldots \end{aligned}$$

4.2. The $\sqrt[3]{3}$ and some multiples of the form $3^{\ell-2/3}$.

Corollary 4.5. *If* $\ell \in \mathbb{Z}^+$, *then*

$$3^{\ell-\frac{2}{3}} = \prod_{k=0}^{\infty}\left(\frac{9k+3\ell+1}{9k+3}\right)\left(\frac{9k+3\ell+4}{9k+6}\right)\left(\frac{9k+3\ell+7}{9k+9\ell+3}\right). \tag{4.3}$$

Proof. Set $n=3$ in Theorem 3.1 and find

$$\begin{aligned} 3^{\ell-1}\cdot\sqrt[3]{3} &= \frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{\ell}{3}+\frac{1}{9}\right)}\prod_{k=1}^{2}\frac{\Gamma\left(\frac{k}{3}\right)}{\Gamma\left(\frac{k}{3}+\frac{\ell}{3}+\frac{1}{9}\right)} \\ \Rightarrow 3^{\ell-\frac{2}{3}} &= \frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{\ell}{3}+\frac{1}{9}\right)}\cdot\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{\ell}{3}+\frac{4}{9}\right)}\cdot\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{\ell}{3}+\frac{7}{9}\right)} \end{aligned} \tag{4.4}$$

Note that

$$\ell+\frac{1}{3}+\frac{1}{3}+\frac{2}{3}=\frac{\ell}{3}+\frac{1}{9}+\frac{\ell}{3}+\frac{4}{9}+\frac{\ell}{3}+\frac{7}{9},$$

satisfy the condition $a_1+a_2+a_3=b_1+b_2+b_3$; $k=3$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and (4.4), it follows that

$$\begin{aligned} 3^{\ell-\frac{2}{3}} &= \prod_{k=0}^{\infty}\left(\frac{k+\frac{\ell}{3}+\frac{1}{9}}{k+\frac{1}{3}}\right)\left(\frac{k+\frac{\ell}{3}+\frac{4}{9}}{k+\frac{2}{3}}\right)\left(\frac{k+\frac{\ell}{3}+\frac{7}{9}}{k+\ell+\frac{1}{3}}\right) \\ &= \prod_{k=0}^{\infty}\left(\frac{9k+3\ell+1}{9k+3}\right)\left(\frac{9k+3\ell+4}{9k+6}\right)\left(\frac{9k+3\ell+7}{9k+9\ell+3}\right) \end{aligned}$$

which is the desired result. □

Example 4.6. Let $\ell=1$ in Corollary 4.5 and encounter

$$\begin{aligned} \sqrt[3]{3} &= \prod_{k=0}^{\infty}\left(\frac{9k+4}{9k+3}\right)\left(\frac{9k+7}{9k+6}\right)\left(\frac{9k+10}{9k+12}\right) \\ &= \frac{4}{3}\cdot\frac{7}{6}\cdot\frac{10}{12}\cdot\frac{13}{12}\cdot\frac{16}{15}\cdot\frac{19}{21}\cdot\frac{22}{21}\cdot\frac{25}{24}\cdot\frac{28}{30}\cdot\cdots \end{aligned}$$

Example 4.7. Let $\ell=2$ in Corollary 4.5 and encounter

$$\begin{aligned} 3\sqrt[3]{3} &= \prod_{k=0}^{\infty}\left(\frac{9k+7}{9k+3}\right)\left(\frac{9k+10}{9k+6}\right)\left(\frac{9k+13}{9k+21}\right) \\ &= \frac{7}{3}\cdot\frac{10}{6}\cdot\frac{13}{21}\cdot\frac{16}{12}\cdot\frac{19}{15}\cdot\frac{22}{30}\cdot\frac{25}{21}\cdot\frac{28}{24}\cdot\frac{31}{39}\cdot\cdots \end{aligned}$$

Example 4.8. Let $\ell=3$ in Corollary 4.5 and encounter

$$\begin{aligned} 9\sqrt[3]{3} &= \prod_{k=0}^{\infty}\left(\frac{9k+10}{9k+3}\right)\left(\frac{9k+13}{9k+6}\right)\left(\frac{9k+16}{9k+30}\right) \\ &= \frac{10}{3}\cdot\frac{13}{6}\cdot\frac{16}{30}\cdot\frac{19}{12}\cdot\frac{22}{15}\cdot\frac{25}{39}\cdot\frac{28}{21}\cdot\frac{31}{24}\cdot\frac{34}{48}\cdot\cdots \end{aligned}$$

4.3. The $\sqrt[4]{4}$ and some multiples of the form $4^{\ell-3/4}$.

Corollary 4.9. *If* $\ell \in \mathbb{Z}^+$, *then*

$$4^{\ell-3/4} = 2^{2\ell-3/2} = \\ = \prod_{k=0}^{\infty} \left(\frac{16k+4\ell+1}{16k+4}\right)\left(\frac{16k+4\ell+5}{16k+8}\right)\left(\frac{16k+4\ell+9}{16k+12}\right)\left(\frac{16k+4\ell+13}{16k+16\ell+4}\right). \tag{4.5}$$

Proof. Set $n=4$ in Theorem 3.1 and find

$$\begin{aligned} 4^{\ell-1}\cdot\sqrt[4]{4} &= \frac{\Gamma\left(\ell+\frac{1}{4}\right)}{\Gamma\left(\frac{\ell}{4}+\frac{1}{16}\right)} \prod_{k=1}^{3} \frac{\Gamma\left(\frac{k}{4}\right)}{\Gamma\left(\frac{k}{4}+\frac{\ell}{4}+\frac{1}{16}\right)} \\ \Rightarrow 4^{\ell-3/4} &= \frac{\Gamma\left(\ell+\frac{1}{4}\right)}{\Gamma\left(\frac{\ell}{4}+\frac{1}{16}\right)} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{\ell}{4}+\frac{5}{16}\right)} \cdot \frac{\Gamma\left(\frac{2}{4}\right)}{\Gamma\left(\frac{\ell}{4}+\frac{9}{16}\right)} \cdot \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{\ell}{4}+\frac{13}{16}\right)} \end{aligned} \tag{4.6}$$

Note that

$$\ell+\frac{1}{4}+\frac{1}{4}+\frac{2}{4}+\frac{3}{4} = \frac{\ell}{4}+\frac{1}{16}+\frac{\ell}{4}+\frac{5}{16}+\frac{\ell}{4}+\frac{9}{16}+\frac{\ell}{4}+\frac{13}{16},$$

satisfy the condition $a_1+a_2+a_3+a_4 = b_1+b_2+b_3+b_4$; $k=4$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and (4.6), it follows that

$$4^{\ell-3/4} = \prod_{k=0}^{\infty} \left(\frac{k+\frac{\ell}{4}+\frac{5}{16}}{k+\frac{1}{4}}\right)\left(\frac{k+\frac{\ell}{4}+\frac{9}{16}}{k+\frac{2}{4}}\right)\left(\frac{k+\frac{\ell}{4}+\frac{13}{16}}{k+\frac{3}{4}}\right)\left(\frac{k+\frac{\ell}{4}+\frac{1}{16}}{k+\ell+\frac{1}{4}}\right)$$

$$\Rightarrow 4^{\ell-3/4} = 2^{2\ell-3/2} = \\ = \prod_{k=0}^{\infty} \left(\frac{16k+4\ell+1}{16k+4}\right)\left(\frac{16k+4\ell+5}{16k+8}\right)\left(\frac{16k+4\ell+9}{16k+12}\right)\left(\frac{16k+4\ell+13}{16k+16\ell+4}\right),$$

which is the desired result. □

Example 4.10. Let $\ell=1$ in Corollary 4.9 and encounter

$$\begin{aligned} \sqrt[4]{4} = \sqrt{2} &= \prod_{k=0}^{\infty} \left(\frac{16k+5}{16k+4}\right)\left(\frac{16k+9}{16k+8}\right)\left(\frac{16k+13}{16k+12}\right)\left(\frac{16k+17}{16k+20}\right) \\ &= \frac{5}{4}\cdot\frac{9}{8}\cdot\frac{13}{12}\cdot\frac{17}{20}\cdot\frac{21}{20}\cdot\frac{25}{24}\cdot\frac{29}{28}\cdot\frac{33}{36}\cdot\dots \end{aligned}$$

Example 4.11. Let $\ell=2$ in Corollary 4.9 and encounter

$$\begin{aligned} 4\sqrt[4]{4} = 4\sqrt{2} &= \prod_{k=0}^{\infty} \left(\frac{16k+9}{16k+4}\right)\left(\frac{16k+13}{16k+8}\right)\left(\frac{16k+17}{16k+12}\right)\left(\frac{16k+21}{16k+36}\right) \\ &= \frac{9}{4}\cdot\frac{13}{8}\cdot\frac{17}{12}\cdot\frac{21}{36}\cdot\frac{25}{20}\cdot\frac{29}{24}\cdot\frac{33}{28}\cdot\frac{38}{52}\cdot\dots \end{aligned}$$

Example 4.12. Let $\ell=3$ in Corollary 4.9 and encounter

$$\begin{aligned} 16\sqrt[4]{4} = 16\sqrt{2} &= \prod_{k=0}^{\infty} \left(\frac{16k+13}{16k+4}\right)\left(\frac{16k+17}{16k+8}\right)\left(\frac{16k+21}{16k+12}\right)\left(\frac{16k+25}{16k+52}\right) \\ &= \frac{13}{4}\cdot\frac{17}{8}\cdot\frac{21}{12}\cdot\frac{25}{52}\cdot\frac{29}{20}\cdot\frac{33}{24}\cdot\frac{37}{28}\cdot\frac{41}{68}\cdot\dots \end{aligned}$$

4.4. The $\sqrt[5]{5}$ and some multiples of the form $5^{\ell-4/5}$.

Corollary 4.13. *If* $\ell\in\mathbb{Z}^+$, *then*

$$5^{\ell-4/5}=\prod_{k=0}^{\infty}\left(\frac{25k+5\ell+1}{25k+25\ell+5}\right)\left(\frac{25k+5\ell+6}{25k+5}\right)\left(\frac{25k+5\ell+11}{25k+10}\right)\left(\frac{25k+5\ell+16}{25k+15}\right)\left(\frac{25k+5\ell+21}{25k+20}\right). \tag{4.7}$$

Proof. Set $n=5$ in Theorem 3.1 and find

$$\begin{aligned}5^{\ell-1}\cdot\sqrt[5]{5}&=\frac{\Gamma\left(\ell+\frac{1}{5}\right)}{\Gamma\left(\frac{\ell}{5}+\frac{1}{25}\right)}\prod_{k=1}^{4}\frac{\Gamma\left(\frac{k}{5}\right)}{\Gamma\left(\frac{k}{5}+\frac{\ell}{5}+\frac{1}{25}\right)}\\ \Rightarrow 5^{\ell-4/5}&=\frac{\Gamma\left(\ell+\frac{1}{5}\right)}{\Gamma\left(\frac{\ell}{5}+\frac{1}{25}\right)}\cdot\frac{\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{\ell}{5}+\frac{6}{25}\right)}\cdot\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{\ell}{5}+\frac{11}{25}\right)}\cdot\frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{\ell}{5}+\frac{16}{25}\right)}\cdot\frac{\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{\ell}{5}+\frac{21}{25}\right)}\end{aligned} \tag{4.8}$$

Note that

$$\ell+\frac{1}{5}+\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}=\frac{\ell}{5}+\frac{1}{25}+\frac{\ell}{5}+\frac{6}{25}+\frac{\ell}{5}+\frac{11}{25}+\frac{\ell}{5}+\frac{16}{25}+\frac{\ell}{5}+\frac{21}{25},$$

satisfy the condition $a_1+a_2+a_3+a_4+a_5=b_1+b_2+b_3+b_4+b_5$; $k=5$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and (4.8), it follows that

$$\begin{aligned}5^{\ell-4/5}&=\prod_{k=0}^{\infty}\left(\frac{k+\frac{\ell}{5}+\frac{1}{25}}{k+\ell+\frac{1}{5}}\right)\left(\frac{k+\frac{\ell}{5}+\frac{6}{25}}{k+\frac{1}{5}}\right)\left(\frac{k+\frac{\ell}{5}+\frac{11}{25}}{k+\frac{2}{5}}\right)\left(\frac{k+\frac{\ell}{5}+\frac{16}{25}}{k+\frac{3}{5}}\right)\left(\frac{k+\frac{\ell}{5}+\frac{21}{25}}{k+\frac{4}{5}}\right)\\ &=\prod_{k=0}^{\infty}\left(\frac{25k+5\ell+1}{25k+25\ell+5}\right)\left(\frac{25k+5\ell+6}{25k+5}\right)\left(\frac{25k+5\ell+11}{25k+10}\right)\\ &\qquad\left(\frac{25k+5\ell+16}{25k+15}\right)\left(\frac{25k+5\ell+21}{25k+20}\right),\end{aligned}$$

which is the desired result. □

Example 4.14. Let $\ell=1$ in Corollary 4.13 and encounter

$$\begin{aligned}\sqrt[5]{5}&=\prod_{k=0}^{\infty}\left(\frac{25k+6}{25k+5}\right)\left(\frac{25k+11}{25k+10}\right)\left(\frac{25k+16}{25k+15}\right)\left(\frac{25k+21}{25k+20}\right)\left(\frac{25k+26}{25k+30}\right)\\ &=\frac{6}{5}\cdot\frac{11}{10}\cdot\frac{16}{15}\cdot\frac{21}{20}\cdot\frac{26}{30}\cdot\frac{31}{30}\cdot\frac{36}{35}\cdot\frac{41}{40}\cdot\frac{46}{45}\cdot\frac{51}{55}\cdots\end{aligned}$$

Example 4.15. Let $\ell=2$ in Corollary 4.13 and encounter

$$\begin{aligned}5\sqrt[5]{5}&=\prod_{k=0}^{\infty}\left(\frac{25k+11}{25k+5}\right)\left(\frac{25k+16}{25k+10}\right)\left(\frac{25k+21}{25k+15}\right)\left(\frac{25k+26}{25k+20}\right)\left(\frac{25k+31}{25k+55}\right)\\ &=\frac{11}{5}\cdot\frac{16}{10}\cdot\frac{21}{15}\cdot\frac{26}{20}\cdot\frac{31}{55}\cdot\frac{36}{30}\cdot\frac{41}{35}\cdot\frac{46}{40}\cdot\frac{51}{45}\cdot\frac{56}{80}\cdots\end{aligned}$$

Example 4.16. Let $\ell=3$ in Corollary 4.13 and encounter

$$25\sqrt[5]{5}=\prod_{k=0}^{\infty}\left(\frac{25k+16}{25k+5}\right)\left(\frac{25k+21}{25k+10}\right)\left(\frac{25k+26}{25k+15}\right)\left(\frac{25k+31}{25k+20}\right)\left(\frac{25k+36}{25k+80}\right)$$
$$=\frac{16}{5}\cdot\frac{21}{10}\cdot\frac{26}{15}\cdot\frac{31}{20}\cdot\frac{36}{80}\cdot\frac{41}{30}\cdot\frac{46}{35}\cdot\frac{51}{40}\cdot\frac{56}{45}\cdot\frac{61}{105}\cdot\ldots$$

5. Exercises

Exercise 5.1. Using the Theorem 3.1 and Corollary 2.1, prove that

$$6^{\ell-5/6}=\prod_{k=0}^{\infty}\left(\frac{36k+6\ell+1}{36k+36\ell+6}\right)\left(\frac{36k+6\ell+7}{36k+6}\right)\left(\frac{36k+6\ell+13}{36k+12}\right)\left(\frac{36k+6\ell+19}{36k+18}\right)\left(\frac{36k+6\ell+25}{36k+24}\right)\left(\frac{36k+6\ell+31}{36k+30}\right), \tag{5.1}$$

$$7^{\ell-6/7}=\prod_{k=0}^{\infty}\left(\frac{49k+7\ell+1}{49k+49\ell+7}\right)\left(\frac{49k+7\ell+8}{49k+7}\right)\left(\frac{49k+7\ell+15}{49k+14}\right)\left(\frac{49k+7\ell+22}{49k+21}\right)\left(\frac{49k+7\ell+29}{49k+28}\right)\left(\frac{49k+7\ell+36}{49k+35}\right)\left(\frac{49k+7\ell+43}{49k+42}\right), \tag{5.2}$$

$$8^{\ell-7/8}=2^{3\ell-21/8}=\prod_{k=0}^{\infty}\left(\frac{64k+8\ell+1}{64k+64\ell+8}\right)\left(\frac{64k+8\ell+9}{64k+8}\right)\left(\frac{64k+8\ell+17}{64k+16}\right)\left(\frac{64k+8\ell+25}{64k+24}\right)\left(\frac{64k+8\ell+33}{64k+32}\right)\left(\frac{64k+8\ell+41}{64k+40}\right)\left(\frac{64k+8\ell+49}{64k+48}\right)\left(\frac{64k+8\ell+57}{64k+56}\right), \tag{5.3}$$

$$9^{\ell-8/9}=\prod_{k=0}^{\infty}\left(\frac{81k+9\ell+1}{81k+81\ell+9}\right)\left(\frac{81k+9\ell+10}{81k+9}\right)\left(\frac{81k+9\ell+19}{81k+18}\right)\left(\frac{81k+9\ell+28}{81k+27}\right)\left(\frac{81k+9\ell+37}{81k+36}\right)\left(\frac{81k+9\ell+46}{81k+45}\right)\left(\frac{81k+9\ell+55}{81k+54}\right)\left(\frac{81k+9\ell+64}{81k+63}\right)\left(\frac{81k+9\ell+73}{81k+72}\right), \tag{5.4}$$

$$10^{\ell-9/10}=\prod_{k=0}^{\infty}\left(\frac{100k+10\ell+1}{100k+100\ell+10}\right)\left(\frac{100k+10\ell+11}{100k+10}\right)\left(\frac{100k+10\ell+21}{100k+20}\right)\left(\frac{100k+10\ell+31}{100k+30}\right)\left(\frac{100k+10\ell+41}{100k+40}\right)\left(\frac{100k+10\ell+51}{100k+50}\right)\left(\frac{100k+10\ell+61}{100k+60}\right)\left(\frac{100k+10\ell+71}{100k+70}\right)\left(\frac{100k+10\ell+81}{100k+80}\right)\left(\frac{100k+10\ell+91}{100k+90}\right) \tag{5.5}$$

and

$$11^{\ell-10/11}=\prod_{k=0}^{\infty}\left(\frac{121k+11\ell+1}{121k+121\ell+11}\right)\left(\frac{121k+11\ell+12}{121k+11}\right)\left(\frac{121k+11\ell+23}{121k+22}\right)\left(\frac{121k+11\ell+34}{121k+33}\right)\left(\frac{121k+11\ell+45}{121k+44}\right)\left(\frac{121k+11\ell+56}{121k+55}\right)\left(\frac{121k+11\ell+67}{121k+66}\right)\left(\frac{121k+11\ell+78}{121k+77}\right)\left(\frac{121k+11\ell+89}{121k+88}\right)\left(\frac{121k+11\ell+100}{121k+99}\right)\left(\frac{121k+11\ell+111}{121k+110}\right) \tag{5.6}$$

Exercise 5.2. Prove that

$$\sqrt[6]{6}=\prod_{k=0}^{\infty}\left(\frac{36k+7}{36k+6}\right)\left(\frac{36k+13}{36k+12}\right)\left(\frac{36k+19}{36k+18}\right)\left(\frac{36k+25}{36k+24}\right)\left(\frac{36k+31}{36k+30}\right)\left(\frac{36k+37}{36k+42}\right)=\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{19}{18}\cdot\frac{25}{24}\cdot\frac{31}{30}\cdot\frac{37}{42}\cdot\ldots, \tag{5.7}$$

$$\sqrt[7]{7}=\prod_{k=0}^{\infty}\left(\frac{49k+8}{49k+7}\right)\left(\frac{49k+15}{49k+14}\right)\left(\frac{49k+22}{49k+21}\right)\left(\frac{49k+29}{49k+28}\right)\left(\frac{49k+36}{49k+35}\right)\left(\frac{49k+43}{49k+42}\right)\left(\frac{49k+50}{49k+56}\right)=\frac{8}{7}\cdot\frac{15}{14}\cdot\frac{22}{21}\cdot\frac{29}{28}\cdot\frac{36}{35}\cdot\frac{43}{42}\cdot\frac{50}{56}\cdot\ldots, \tag{5.8}$$

$$\sqrt[8]{8}=\prod_{k=0}^{\infty}\left(\frac{64k+9}{64k+8}\right)\left(\frac{64k+17}{64k+16}\right)\left(\frac{64k+25}{64k+24}\right)\left(\frac{64k+33}{64k+32}\right)\left(\frac{64k+41}{64k+40}\right)\left(\frac{64k+49}{64k+48}\right)\left(\frac{64k+57}{64k+56}\right)\left(\frac{64k+65}{64k+72}\right)=\frac{9}{8}\cdot\frac{17}{16}\cdot\frac{25}{24}\cdot\frac{33}{32}\cdot\frac{41}{40}\cdot\frac{49}{48}\cdot\frac{57}{56}\cdot\frac{65}{72}\cdot\ldots, \tag{5.9}$$

$$\sqrt[9]{9}=\prod_{k=0}^{\infty}\left(\frac{81k+10}{81k+9}\right)\left(\frac{81k+19}{81k+18}\right)\left(\frac{81k+28}{81k+27}\right)\left(\frac{81k+37}{81k+36}\right)\left(\frac{81k+46}{81k+45}\right)\left(\frac{81k+55}{81k+54}\right)\left(\frac{81k+64}{81k+63}\right)\left(\frac{81k+73}{81k+72}\right)\left(\frac{81k+82}{81k+90}\right)=\frac{10}{9}\cdot\frac{19}{18}\cdot\frac{28}{27}\cdot\frac{37}{36}\cdot\frac{46}{45}\cdot\frac{55}{54}\cdot\frac{64}{63}\cdot\frac{73}{72}\cdot\frac{82}{90}\cdot\ldots, \tag{5.10}$$

$$\sqrt[10]{10}=\prod_{k=0}^{\infty}\left(\frac{100k+11}{100k+10}\right)\left(\frac{100k+21}{100k+20}\right)\left(\frac{100k+31}{100k+30}\right)\left(\frac{100k+41}{100k+40}\right)\left(\frac{100k+51}{100k+50}\right)\left(\frac{100k+61}{100k+60}\right)\left(\frac{100k+71}{100k+70}\right)\left(\frac{100k+81}{100k+80}\right)\left(\frac{100k+91}{100k+90}\right)\left(\frac{100k+101}{100k+110}\right)=\frac{11}{10}\cdot\frac{21}{20}\cdot\frac{31}{30}\cdot\frac{41}{40}\cdot\frac{51}{50}\cdot\frac{61}{60}\cdot\frac{71}{70}\cdot\frac{81}{80}\cdot\frac{91}{90}\cdot\frac{101}{110}\cdot\ldots \tag{5.11}$$

and

$$\sqrt[11]{11}=\prod_{k=0}^{\infty}\left(\frac{121k+12}{121k+11}\right)\left(\frac{121k+23}{121k+22}\right)\left(\frac{121k+34}{121k+33}\right)\left(\frac{121k+45}{121k+44}\right)\left(\frac{121k+56}{121k+55}\right)\left(\frac{121k+67}{121k+66}\right)\left(\frac{121k+78}{121k+77}\right)\left(\frac{121k+89}{121k+88}\right)\left(\frac{121k+100}{121k+99}\right)\left(\frac{121k+111}{121k+110}\right)\left(\frac{121k+122}{121k+132}\right)=\frac{12}{11}\cdot\frac{23}{22}\cdot\frac{34}{33}\cdot\frac{45}{44}\cdot\frac{56}{55}\cdot\frac{67}{66}\cdot\frac{78}{77}\cdot\frac{89}{88}\cdot\frac{100}{99}\cdot\frac{111}{110}\cdot\frac{122}{132}\cdot\ldots \tag{5.12}$$

6. Some Logarithm Constants and Infinite Sum Representation

6.1. The $\ln 2/2$.

Corollary 6.1. *If $\ell \in \mathbb{Z}^+$, then*

$$\frac{\ln 2}{2} = \sum_{k=0}^{\infty} \frac{4\ell^2 + (8k+4)\ell + 1}{(2k+2\ell+1)(4k+2\ell+1)(4k+2\ell+3)}. \tag{6.1}$$

Proof. The logarihtmic differentiation of the Corollary 4.1 with respect to ℓ give me the desired result. □

Example 6.2. Let $\ell = 1$ in Corollary 6.1 and encounter

$$\frac{\ln 2}{2} = \sum_{k=0}^{\infty} \frac{8k+9}{(2k+3)(4k+3)(4k+5)}$$
$$= \frac{9}{3 \cdot 3 \cdot 5} + \frac{17}{5 \cdot 7 \cdot 9} + \frac{25}{7 \cdot 11 \cdot 13} + \dots$$

Example 6.3. Let $\ell = 2$ in Corollary 6.1 and encounter

$$\frac{\ln 2}{2} = \sum_{k=0}^{\infty} \frac{16k+25}{(2k+5)(4k+5)(4k+7)}$$
$$= \frac{25}{5 \cdot 5 \cdot 7} + \frac{41}{7 \cdot 9 \cdot 11} + \frac{57}{9 \cdot 13 \cdot 15} + \dots$$

6.2. The $\ln 3/3$.

Corollary 6.4. *If $\ell \in \mathbb{Z}^+$, then*

$$\frac{\ln 3}{3} = \sum_{k=0}^{\infty} \frac{81k^2(6\ell-1) + 18k(18\ell^2+21\ell-1) + (6\ell+11)(3\ell+1)^2}{(3k+3\ell+1)(9k+3\ell+1)(9k+3\ell+4)(9k+3\ell+7)}. \tag{6.2}$$

Proof. The logarihtmic differentiation of the Corollary 4.5 with respect to ℓ give me the desired result. □

Example 6.5. Let $\ell = 1$ in Corollary 6.4 and encounter

$$\frac{\ln 3}{3} = \sum_{k=0}^{\infty} \frac{405k^2 + 684k + 272}{(3k+4)(9k+4)(9k+7)(9k+10)}$$
$$= \frac{272}{4 \cdot 4 \cdot 7 \cdot 10} + \frac{1361}{7 \cdot 13 \cdot 16 \cdot 19} + \frac{3260}{10 \cdot 22 \cdot 25 \cdot 28} + \dots$$

Example 6.6. Let $\ell = 2$ in Corollary 6.4 and encounter

$$\frac{\ln 3}{3} = \sum_{k=0}^{\infty} \frac{891k^2 + 2034k + 1127}{(3k+7)(9k+7)(9k+10)(9k+13)}$$
$$= \frac{1127}{7 \cdot 7 \cdot 10 \cdot 13} + \frac{4052}{10 \cdot 16 \cdot 19 \cdot 22} + \frac{8759}{13 \cdot 25 \cdot 28 \cdot 31} + \dots$$

7. APPLICATION FOR THE DIVISION BETWEEN GAMMA FUNCTIONS

Theorem 7.1. *If $a \in \mathbb{Z}_{\geqslant 2}$, $b \in \mathbb{Z}_{\geqslant 3}$, $a < b$ and $k \in \mathbb{N}$, then*

$$\left(\frac{a}{b}\right)^k = \frac{a\sqrt[b]{b}}{b\sqrt[a]{a}} \cdot \frac{\Gamma\left(k+\frac{1}{a}\right)\Gamma\left(\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(k+\frac{1}{b}\right)\Gamma\left(\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \prod_{\nu=1}^{a-1} \frac{\Gamma\left(\frac{\nu}{a}\right)}{\Gamma\left(\frac{\nu}{a}+\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \prod_{\nu=1}^{b-1} \frac{\Gamma\left(\frac{\nu}{b}+\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(\frac{\nu}{b}\right)},$$

where $\Gamma(z)$ denotes the gamma function.

Proof. In Theorem 3.1, replace n by a, ℓ by k and k by ν

$$a^{k-1} \cdot \sqrt[a]{a} = \frac{\Gamma\left(k+\frac{1}{a}\right)}{\Gamma\left(\frac{k}{a}+\frac{1}{a^2}\right)} \prod_{\nu=1}^{a-1} \frac{\Gamma\left(\frac{\nu}{a}\right)}{\Gamma\left(\frac{\nu}{a}+\frac{k}{a}+\frac{1}{a^2}\right)}. \tag{7.1}$$

In (7.2), replace a by b

$$b^{k-1} \cdot \sqrt[b]{b} = \frac{\Gamma\left(k+\frac{1}{b}\right)}{\Gamma\left(\frac{k}{b}+\frac{1}{b^2}\right)} \prod_{\nu=1}^{b-1} \frac{\Gamma\left(\frac{\nu}{b}\right)}{\Gamma\left(\frac{\nu}{b}+\frac{k}{b}+\frac{1}{b^2}\right)}. \tag{7.2}$$

Divide (7.2) by (7.3) and find

$$\left(\frac{a}{b}\right)^k \cdot \frac{b\sqrt[a]{a}}{a\sqrt[b]{b}} = \frac{\Gamma\left(k+\frac{1}{a}\right)\Gamma\left(\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(k+\frac{1}{b}\right)\Gamma\left(\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \frac{\prod_{\nu=1}^{a-1} \frac{\Gamma\left(\frac{\nu}{a}\right)}{\Gamma\left(\frac{\nu}{a}+\frac{k}{a}+\frac{1}{a^2}\right)}}{\prod_{\nu=1}^{b-1} \frac{\Gamma\left(\frac{\nu}{b}\right)}{\Gamma\left(\frac{\nu}{b}+\frac{k}{b}+\frac{1}{b^2}\right)}}$$

$$\Rightarrow \left(\frac{a}{b}\right)^k = \frac{a\sqrt[b]{b}}{b\sqrt[a]{a}} \cdot \frac{\Gamma\left(k+\frac{1}{a}\right)\Gamma\left(\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(k+\frac{1}{b}\right)\Gamma\left(\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \frac{\frac{\prod_{\nu=1}^{a-1}\Gamma\left(\frac{\nu}{a}\right)}{\prod_{\nu=1}^{a-1}\Gamma\left(\frac{\nu}{a}+\frac{k}{a}+\frac{1}{a^2}\right)}}{\frac{\prod_{\nu=1}^{b-1}\Gamma\left(\frac{\nu}{b}\right)}{\prod_{\nu=1}^{b-1}\Gamma\left(\frac{\nu}{b}+\frac{k}{b}+\frac{1}{b^2}\right)}}$$

$$\Rightarrow \left(\frac{a}{b}\right)^k = \frac{a\sqrt[b]{b}}{b\sqrt[a]{a}} \cdot \frac{\Gamma\left(k+\frac{1}{a}\right)\Gamma\left(\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(k+\frac{1}{b}\right)\Gamma\left(\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \frac{\prod_{\nu=1}^{a-1}\Gamma\left(\frac{\nu}{a}\right)\prod_{\nu=1}^{b-1}\Gamma\left(\frac{\nu}{b}+\frac{k}{b}+\frac{1}{b^2}\right)}{\prod_{\nu=1}^{b-1}\Gamma\left(\frac{\nu}{b}\right)\prod_{\nu=1}^{a-1}\Gamma\left(\frac{\nu}{a}+\frac{k}{a}+\frac{1}{a^2}\right)}$$

$$\Rightarrow \left(\frac{a}{b}\right)^k = \frac{a\sqrt[b]{b}}{b\sqrt[a]{a}} \cdot \frac{\Gamma\left(k+\frac{1}{a}\right)\Gamma\left(\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(k+\frac{1}{b}\right)\Gamma\left(\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \prod_{\nu=1}^{a-1} \frac{\Gamma\left(\frac{\nu}{a}\right)}{\Gamma\left(\frac{\nu}{a}+\frac{k}{a}+\frac{1}{a^2}\right)} \cdot \prod_{\nu=1}^{b-1} \frac{\Gamma\left(\frac{\nu}{b}+\frac{k}{b}+\frac{1}{b^2}\right)}{\Gamma\left(\frac{\nu}{b}\right)},$$

which is the desired result. □

Corollary 7.2. *I have*

$$\sqrt[3]{3} = \frac{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)}{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)}. \tag{7.3}$$

Proof. Set $a=2$ and $b=3$ in Theorem 7.1

$$\left(\frac{2}{3}\right)^k=\frac{\sqrt[3]{3}}{\sqrt{6\pi}}\cdot\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{k}{3}+\frac{1}{9}\right)\Gamma\left(\frac{k}{3}+\frac{4}{9}\right)\Gamma\left(\frac{k}{3}+\frac{7}{9}\right)}{\Gamma\left(k+\frac{1}{3}\right)\Gamma\left(\frac{k}{2}+\frac{1}{4}\right)\Gamma\left(\frac{k}{2}+\frac{3}{4}\right)}. \tag{7.4}$$

For $k=1$ in (7.4), I obtain

$$\frac{2\pi\sqrt{3}}{\sqrt[3]{3}}=\frac{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{7}{9}\right)}{\Gamma\left(\frac{1}{3}\right)}. \tag{7.5}$$

Multiply the numerator and the denominator in the right hand side of (7.5) by $\Gamma(2/9)\Gamma(5/9)\Gamma(8/9)$ and apply the Euler's reflection formula for gamma function, i. e., $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$,

$$\begin{aligned}\frac{2\pi\sqrt{3}}{\sqrt[3]{3}}&=\frac{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)}\\ \Rightarrow\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)&=\frac{4\pi^2\sqrt[3]{3}}{3}.\end{aligned} \tag{7.6}$$

Multiply the numerator and the denominator in the right hand side of (7.5) by $\Gamma(2/3)$ and apply the Euler's reflection formula for gamma function, i. e., $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$,

$$\begin{aligned}\frac{2\pi\sqrt{3}}{\sqrt[3]{3}}&=\frac{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{7}{9}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\\ \Rightarrow\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)&=\frac{4\pi^2}{\sqrt[3]{3}}.\end{aligned} \tag{7.7}$$

Divide (7.7) by (7.6) and encounter the desired result. □

Corollary 7.3. *I have*

$$\begin{aligned}\sqrt[3]{3}&=\prod_{k=0}^{\infty}\frac{(9k+2)(9k+3)(9k+5)(9k+8)}{(9k+1)(9k+4)(9k+6)(9k+7)}\\ &=\frac{2}{1}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{8}{7}\cdot\frac{11}{10}\cdot\frac{12}{13}\cdot\frac{14}{15}\cdot\frac{17}{16}\cdot\ldots\end{aligned} \tag{7.8}$$

Proof. I use the Corollary 2.1 in the right hand side of the Corollary 7.2. □

Corollary 7.4. *I have*

$$2=\frac{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{12}{16}\right)\Gamma\left(\frac{13}{16}\right)}{\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{4}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right)}. \tag{7.9}$$

Proof. Set $a=3$ and $b=4$ in Theorem 7.1

$$\left(\frac{3}{4}\right)^k=\frac{3\sqrt[4]{4}}{4\sqrt[3]{3}}\cdot\sqrt{\frac{2}{3\pi}}\cdot\frac{\Gamma\left(k+\frac{1}{3}\right)\Gamma\left(\frac{k}{4}+\frac{1}{16}\right)\Gamma\left(\frac{k}{4}+\frac{5}{16}\right)\Gamma\left(\frac{k}{4}+\frac{9}{16}\right)\Gamma\left(\frac{k}{4}+\frac{13}{16}\right)}{\Gamma\left(k+\frac{1}{4}\right)\Gamma\left(\frac{k}{3}+\frac{1}{9}\right)\Gamma\left(\frac{k}{3}+\frac{4}{9}\right)\Gamma\left(\frac{k}{3}+\frac{7}{9}\right)}. \tag{7.10}$$

For $k=1$ in (7.10), I obtain

$$\frac{4\sqrt[3]{3}}{3\sqrt[4]{4}}\sqrt{\frac{3\pi}{2}}=\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{7}{9})}. \tag{7.11}$$

Multiply the numerator and the denominator in the right hand side of (7.11) by $\Gamma(2/3)\Gamma(15/16)\Gamma(11/16)\Gamma(7/16)\Gamma(3/16)$ and apply the Euler's reflection formula for gamma function, i. e., $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$,

$$\begin{aligned}\frac{4\sqrt[3]{3}}{3\sqrt[4]{4}}\sqrt{\frac{3\pi}{2}}=&\frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{1}{16})\Gamma(\frac{15}{16})\Gamma(\frac{5}{16})\Gamma(\frac{11}{16})\Gamma(\frac{7}{16})\Gamma(\frac{9}{16})\Gamma(\frac{3}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{6}{9})\Gamma(\frac{7}{9})\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}\\ \Rightarrow\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)&\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{4}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right)\\ =&\frac{8\pi^5\sqrt[4]{4}}{\sqrt{\pi}\sqrt[3]{3}}.\end{aligned} \tag{7.12}$$

Multiply the numerator and the denominator in the right hand side of (7.5) by $\Gamma(2/9)\Gamma(5/9)\Gamma(8/9)\Gamma(3/4)$ and apply the Euler's reflection formula for gamma function, i. e., $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$,

$$\begin{aligned}\frac{4\sqrt[3]{3}}{3\sqrt[4]{4}}\sqrt{\frac{3\pi}{2}}=&\frac{\Gamma(\frac{2}{9})\Gamma(\frac{3}{9})\Gamma(\frac{5}{9})\Gamma(\frac{8}{9})\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})\Gamma(\frac{1}{9})\Gamma(\frac{8}{9})\Gamma(\frac{4}{9})\Gamma(\frac{5}{9})\Gamma(\frac{2}{9})\Gamma(\frac{7}{9})}\\ \Rightarrow\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)&\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{12}{16}\right)\Gamma\left(\frac{13}{16}\right)\\ =&\frac{32\pi^4\sqrt{\pi}\sqrt[3]{3}}{3\sqrt[4]{4}}.\end{aligned} \tag{7.13}$$

Divide (7.12) by (7.13) and encounter

$$\frac{\sqrt[3]{3}}{2}=\frac{\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{6}{9})\Gamma(\frac{7}{9})\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}{\Gamma(\frac{2}{9})\Gamma(\frac{3}{9})\Gamma(\frac{5}{9})\Gamma(\frac{8}{9})\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}. \tag{7.14}$$

From Corollary 7.2 and (7.14), it follows that

$$\begin{aligned}\frac{1}{2}=&\frac{\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}{\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}\\ \Rightarrow2=&\frac{\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})},\end{aligned}$$

which is the desired result. □

Corollary 7.5. *I have*

$$\begin{aligned}2=&\prod_{k=0}^{\infty}\frac{(16k+3)(16k+4)(16k+7)(16k+11)(16k+15)}{(16k+1)(16k+5)(16k+9)(16k+12)(16k+13)}\\ =&\frac{3}{1}\cdot\frac{4}{5}\cdot\frac{7}{9}\cdot\frac{11}{12}\cdot\frac{15}{13}\cdot\frac{19}{17}\cdot\frac{20}{21}\cdot\frac{23}{25}\cdot\frac{27}{28}\cdot\frac{31}{29}\cdot\ldots\end{aligned} \tag{7.15}$$

Proof. I use the Corollary 2.1 in the right hand side of the Corollary 7.4. □

8. Exercises

Exercise 8.1. Use the Theorem 7.1 and prove that

$$\sqrt[5]{5^3}=\frac{\Gamma(\frac{1}{25})\Gamma(\frac{6}{25})\Gamma(\frac{11}{25})\Gamma(\frac{16}{25})\Gamma(\frac{20}{25})\Gamma(\frac{21}{25})}{\Gamma(\frac{4}{25})\Gamma(\frac{5}{25})\Gamma(\frac{9}{25})\Gamma(\frac{14}{25})\Gamma(\frac{19}{25})\Gamma(\frac{24}{25})}, \tag{8.1}$$

$$\sqrt[6]{6^4}=\sqrt[3]{6^2}=\frac{\Gamma(\frac{1}{36})\Gamma(\frac{7}{36})\Gamma(\frac{13}{36})\Gamma(\frac{19}{36})\Gamma(\frac{25}{36})\Gamma(\frac{30}{36})\Gamma(\frac{31}{36})}{\Gamma(\frac{5}{36})\Gamma(\frac{6}{36})\Gamma(\frac{11}{36})\Gamma(\frac{17}{36})\Gamma(\frac{23}{36})\Gamma(\frac{29}{36})\Gamma(\frac{35}{36})}, \tag{8.2}$$

$$\sqrt[7]{7^5}=\frac{\Gamma(\frac{1}{49})\Gamma(\frac{8}{49})\Gamma(\frac{15}{49})\Gamma(\frac{22}{49})\Gamma(\frac{29}{49})\Gamma(\frac{36}{49})\Gamma(\frac{42}{49})\Gamma(\frac{43}{49})}{\Gamma(\frac{6}{49})\Gamma(\frac{7}{49})\Gamma(\frac{13}{49})\Gamma(\frac{20}{49})\Gamma(\frac{27}{49})\Gamma(\frac{34}{49})\Gamma(\frac{41}{49})\Gamma(\frac{48}{49})}, \tag{8.3}$$

$$\sqrt[8]{8^6}=4\sqrt[4]{2}=\frac{\Gamma(\frac{1}{64})\Gamma(\frac{9}{64})\Gamma(\frac{17}{64})\Gamma(\frac{25}{64})\Gamma(\frac{33}{64})\Gamma(\frac{41}{64})\Gamma(\frac{49}{64})\Gamma(\frac{56}{64})\Gamma(\frac{57}{64})}{\Gamma(\frac{7}{64})\Gamma(\frac{8}{64})\Gamma(\frac{15}{64})\Gamma(\frac{23}{64})\Gamma(\frac{31}{64})\Gamma(\frac{39}{64})\Gamma(\frac{47}{64})\Gamma(\frac{55}{64})\Gamma(\frac{63}{64})}, \tag{8.4}$$

$$\sqrt[9]{9^7}=\frac{\Gamma(\frac{1}{81})\Gamma(\frac{10}{81})\Gamma(\frac{19}{81})\Gamma(\frac{28}{81})\Gamma(\frac{37}{81})\Gamma(\frac{46}{81})\Gamma(\frac{55}{81})\Gamma(\frac{64}{81})\Gamma(\frac{72}{81})\Gamma(\frac{73}{81})}{\Gamma(\frac{8}{81})\Gamma(\frac{9}{81})\Gamma(\frac{17}{81})\Gamma(\frac{26}{81})\Gamma(\frac{35}{81})\Gamma(\frac{44}{81})\Gamma(\frac{53}{81})\Gamma(\frac{62}{81})\Gamma(\frac{71}{81})\Gamma(\frac{80}{81})}, \tag{8.5}$$

$$\sqrt[10]{10^8}=$$
$$=\frac{\Gamma(\frac{1}{100})\Gamma(\frac{11}{100})\Gamma(\frac{21}{100})\Gamma(\frac{31}{100})\Gamma(\frac{41}{100})\Gamma(\frac{51}{100})\Gamma(\frac{61}{100})\Gamma(\frac{71}{100})\Gamma(\frac{81}{100})\Gamma(\frac{90}{100})\Gamma(\frac{91}{100})}{\Gamma(\frac{9}{100})\Gamma(\frac{10}{100})\Gamma(\frac{19}{100})\Gamma(\frac{29}{100})\Gamma(\frac{39}{100})\Gamma(\frac{49}{100})\Gamma(\frac{59}{100})\Gamma(\frac{69}{100})\Gamma(\frac{79}{100})\Gamma(\frac{89}{100})\Gamma(\frac{99}{100})} \tag{8.6}$$

and so on.

Exercise 8.2. Use the Corollary 2.1 and prove that

$$\sqrt[5]{5^3}=\prod_{k=0}^{\infty}\frac{(25k+4)(25k+5)(25k+9)(25k+14)(25k+19)(25k+24)}{(25k+1)(25k+6)(25k+11)(25k+16)(25k+20)(25k+21)}, \tag{8.7}$$

$$\sqrt[6]{6^4}=\sqrt[3]{6^2}=$$
$$=\prod_{k=0}^{\infty}\frac{(36k+5)(36k+6)(36k+11)(36k+17)(36k+23)(36k+29)(36k+35)}{(36k+1)(36k+7)(36k+13)(36k+19)(36k+25)(36k+30)(36k+31)} \tag{8.8}$$

and so on.

References

[1] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1978.

[2] Sondow,Jonathan and Yi, Huang, *New Wallis- and Catalan-Type Infinite Products for π, e and $\sqrt{2+\sqrt{2}}$*, arXiv:1005.2712.

[3] https://proofwiki.org/wiki/Gauss_Multiplication_Formula.