

Cosine fixed point constant and related sequences

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Abstract: This note presents some sequences related with the cosine fixed point constant.

1. Introduction. The cosine fixed point constant d is defined by

$$d = \lim_{n \rightarrow \infty} \cos^{[n]}(x) = \lim_{n \rightarrow \infty} \underbrace{\cos(\cos(\cos(\dots(\cos x))))}_n = 0.739085\dots \quad (1)$$

The limit (1) is independent of x .

Equation for d :

$$\cos d = d \quad (2)$$

The cosine fixed point constant is the unique real root of $\cos x = x$.

2. Related Sequences.

- Sequence 1. (Convergent)

$$x_{n+1} = \cos x_n, \quad x_0 = 1 \Rightarrow x_n \rightarrow d \quad (3)$$

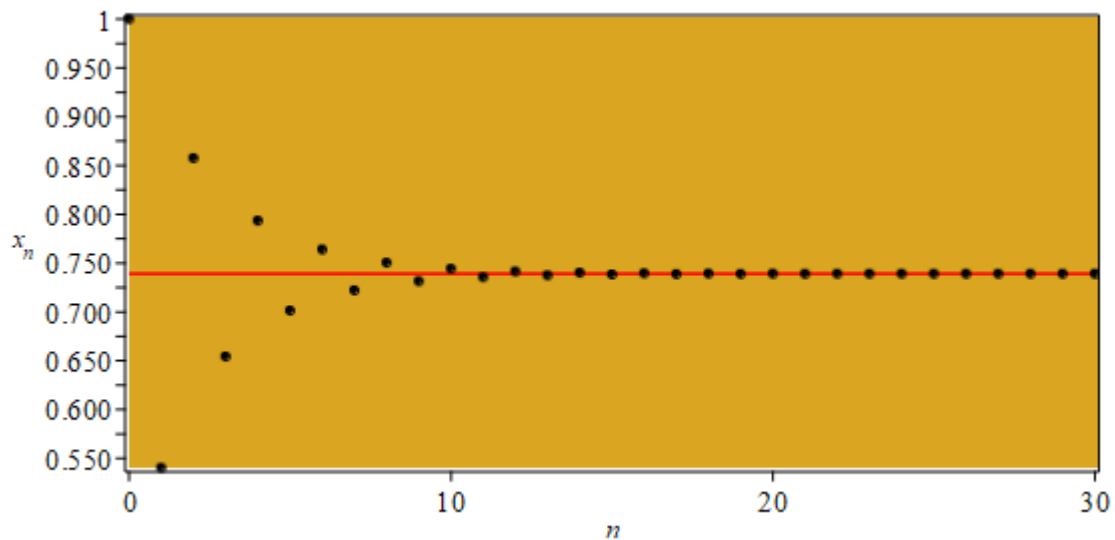


Figure 1. • x_n , • d

- Sequence 2. (Divergent, Oscillation)

$$x_{n+1} = \frac{1}{2} \sin^{-1} \left(2x_n \sqrt{1-x_n^2} \right), \quad 0 < x_0 < 1 \quad (4)$$

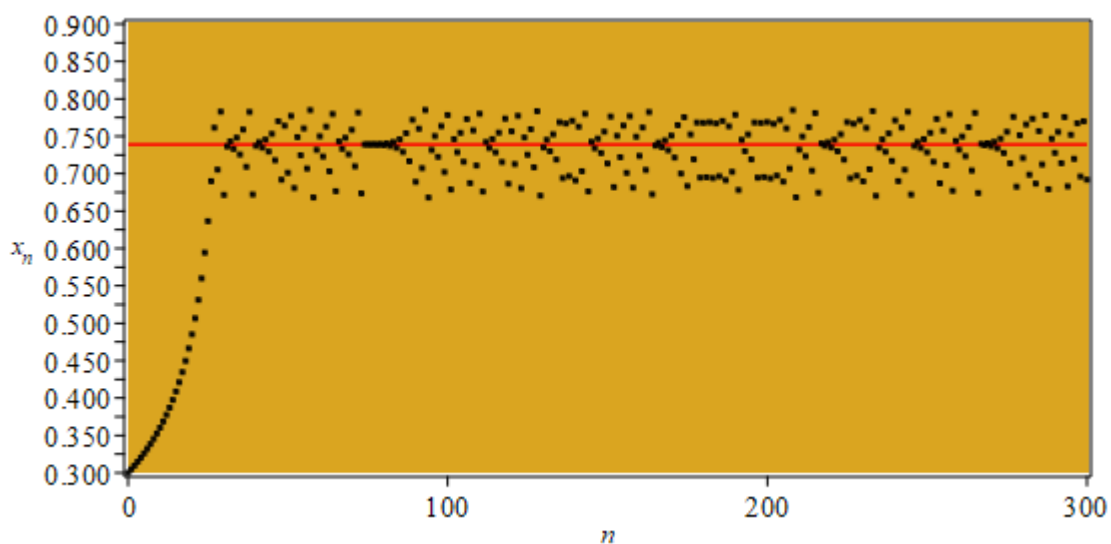


Figure 2. $x_0 = 0.3$, $\bullet x_n$, $\bullet d$

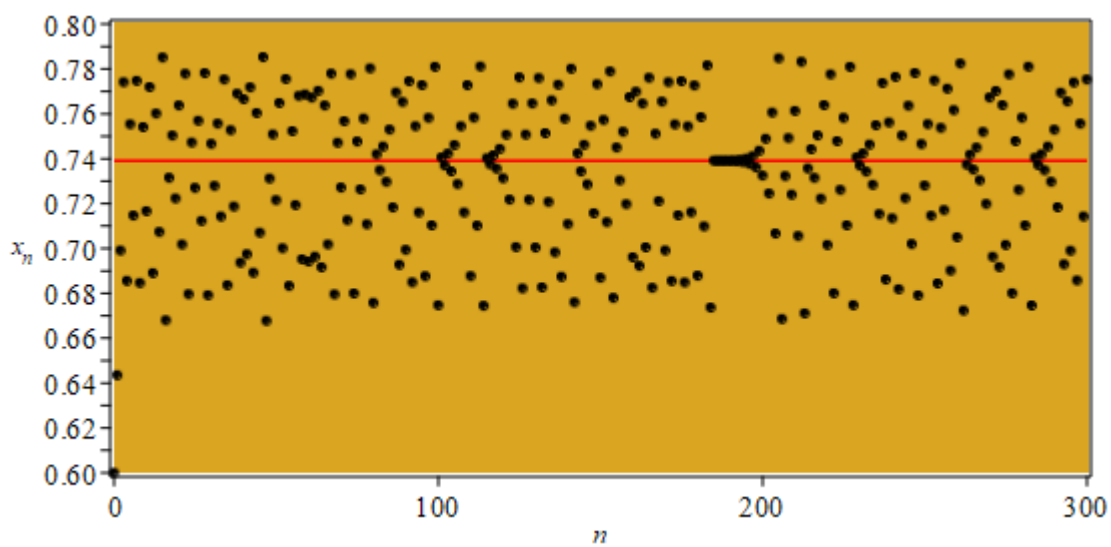


Figure 3. $x_0 = 0.6$, $\bullet x_n$, $\bullet d$

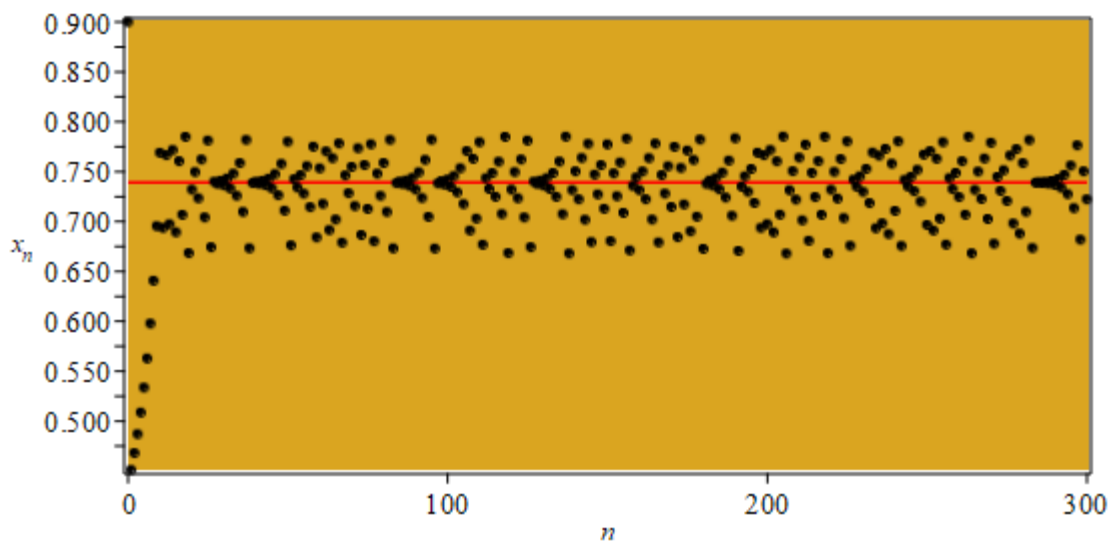


Figure 4. $x_0 = 0.9$, $\bullet x_n$, $\bullet d$

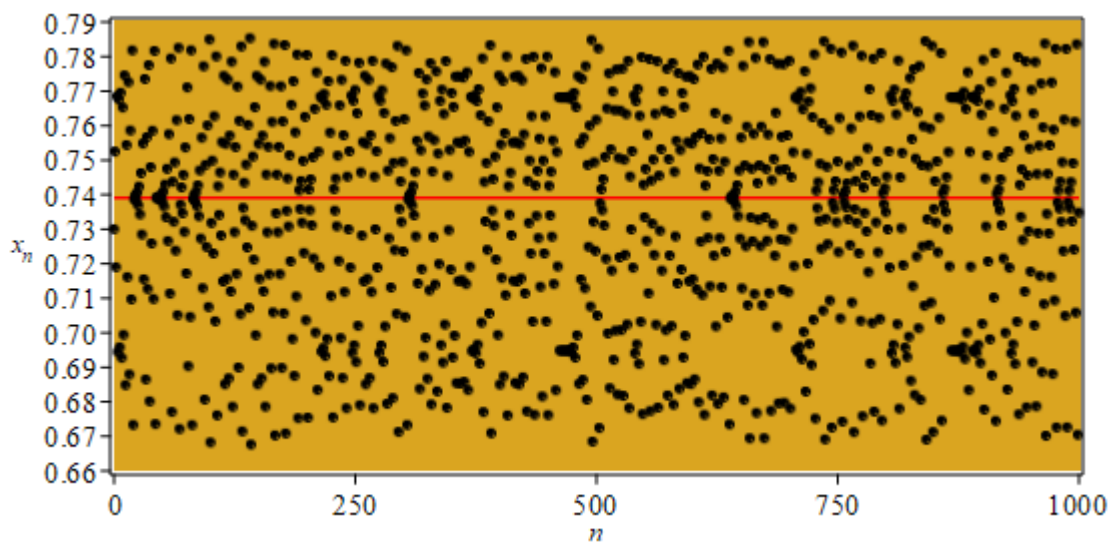


Figure 5. $x_0 = 0.73, n = 1000$, $\bullet x_n$, $\bullet d$

- Sequence 3. (Convergent)

$$x_{n+2} = \frac{1}{2} f(x_{n+1}) + \frac{1}{2} f(x_n) \quad , x_0, x_1 \in (0,1) \Rightarrow x_n \rightarrow d \quad (5)$$

$$f(x) = \frac{1}{2} \sin^{-1}(2x\sqrt{1-x^2}) \quad (6)$$

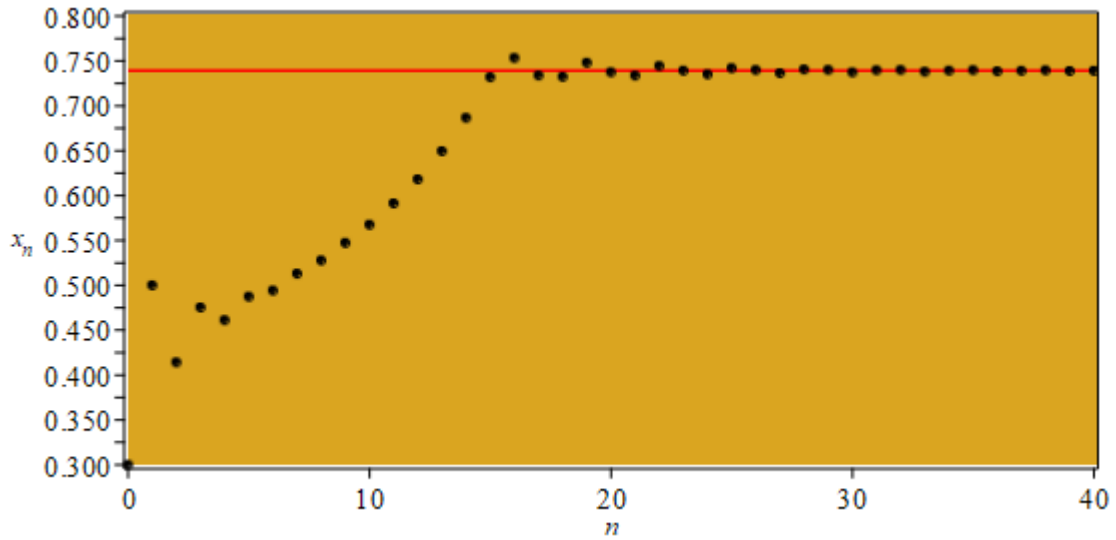


Figure 6. $x_0 = 0.3, x_1 = 0.5, n = 40$, $\bullet x_n$, $\bullet d$

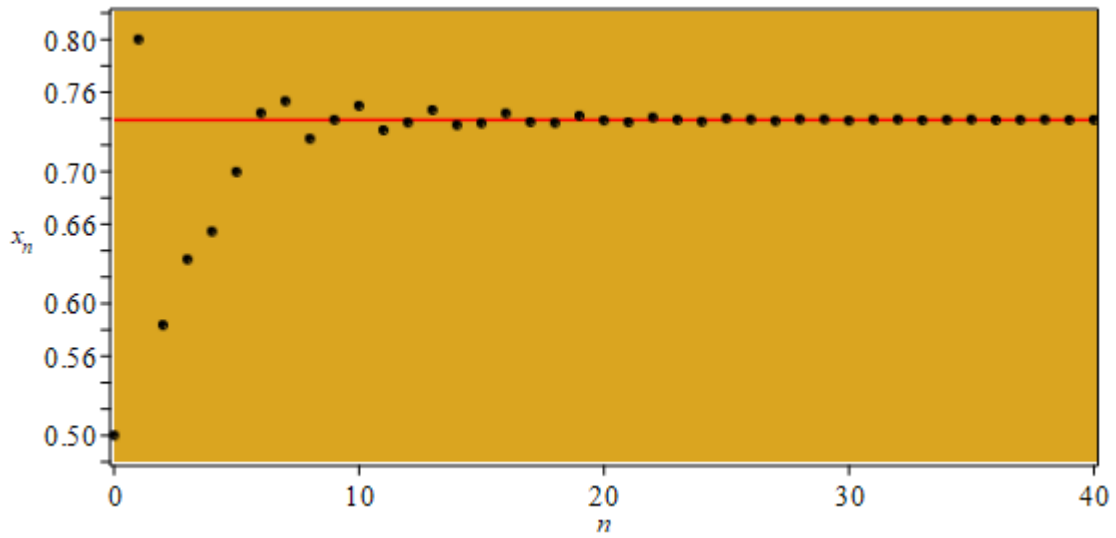


Figure 7. $x_0 = 0.5, x_1 = 0.8, n = 40$, $\bullet x_n$, $\bullet d$

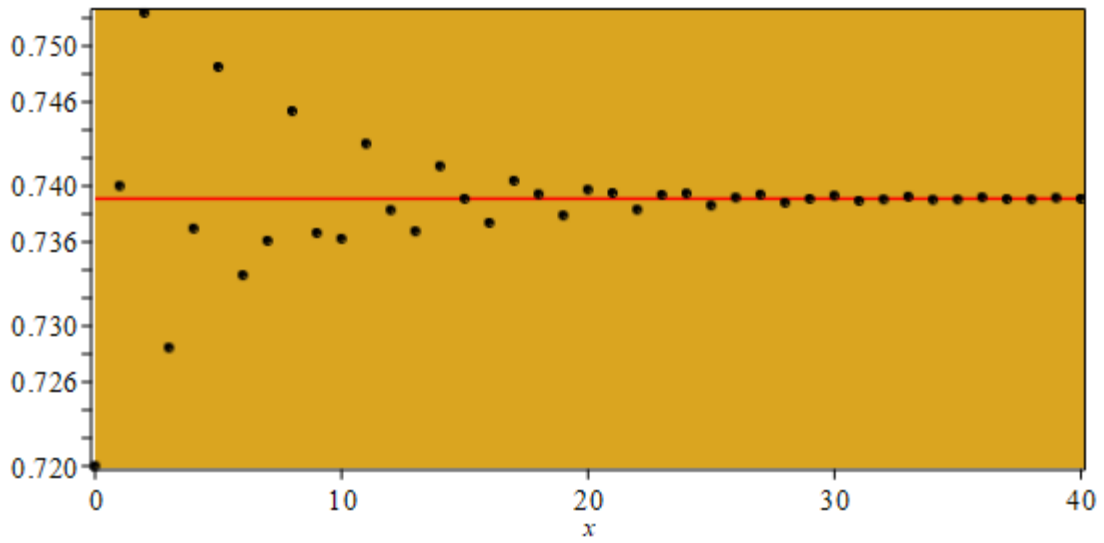


Figure 8. $x_0 = 0.72, x_1 = 0.74, n = 40$, $\bullet x_n$, $\bullet d$

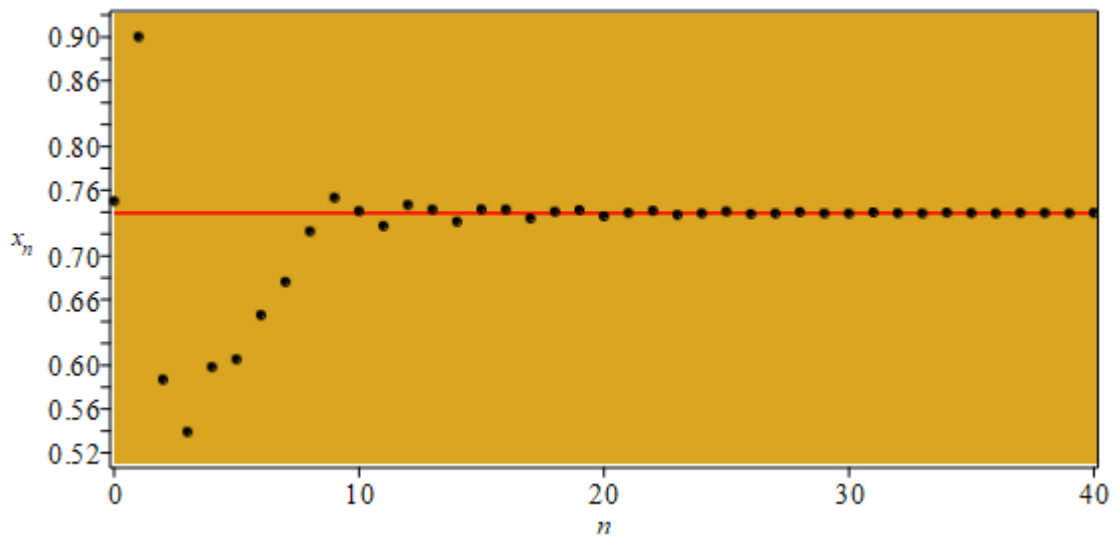


Figure 9. $x_0 = 0.75, x_1 = 0.9, n = 40$, $\bullet x_n$, $\bullet d$

3. Some Remarks:

Let $f(x) = \frac{1}{2} \sin^{-1}(2x\sqrt{1-x^2})$, $x \in [0,1]$, then

- $f([0,1]) \subset [0,1]$
- $|f'(x)| \geq 1$, $x \in [0,1]$

$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}|1-2x^2|} = \frac{\text{sign}(1-2x^2)}{\sqrt{1-x^2}}, \quad 0 \leq x \leq 1 \quad (7)$$

$$\text{sign}(x) = \begin{cases} +1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases} \quad (8)$$

- If $x_0 \in [0,1]$, $x_{n+1} = f(x_n)$, then $0 \leq x_n \leq \pi/4$, $n \in \mathbb{N}$.
- $0 < x_0 < 1$, $x_0 \neq d$, $x_{n+1} = f(x_n) \Rightarrow x_n$ is Divergent (oscillation).
- If $n \in \mathbb{N} \cup \{0\}$, $x_0, x_1 \in (0,1)$, $x_{n+2} = \frac{f(x_{n+1}) + f(x_n)}{2}$, then

$$|x_{n+3} - x_{n+2}| \leq \frac{2}{\sqrt{16-\pi^2}} |x_{n+2} - x_n| \quad (9)$$

- $x, y \in (0,1) \Rightarrow 0 < \frac{f(x) + f(y)}{2} \leq \pi/4$.

- If $n \in \mathbb{N} \cup \{0\}$, $x_0, x_1 \in (0,1)$, $x_{n+2} = \frac{f(x_{n+1}) + f(x_n)}{2}$, then

$$|x_{n+2} - d| \leq \frac{2}{\sqrt{16-\pi^2}} (|x_{n+1} - d| + |x_n - d|), \quad n = 2, 3, 4, \dots \quad (10)$$

References

1. Banks, J., Brooks, J., Cairns, G., Davis, G., Stacey, P.: On Devaney's definition of chaos. Amer. Math. Monthly, Vol. 99, 332-334, 1992.
2. Devaney, R.L.: An introduction to chaotic dynamical systems. Benjamin-Cummings. Menlo Park, 1986.
3. Kaplan, S.: The Dottie Number. Mathematics Magazine, Vol. 80, N.1, 2007.
4. Valdebenito, E.: The Dottie Number. viXra:1802.0427, 2018.
5. Valdebenito, E.: Fixed Point Method, Dottie Number. viXra:1805.0033, 2018.