A NOTE ON THE DIOPHANTINE EQUATION $2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} + 1 = y^p$

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Abstract. In this paper we prove that the diophantine equation $2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} + 1 = y^p, \alpha_1, \alpha_2, \alpha_3, y \in \mathbb{N}$ and $p \geq 2$ is a prime, has only finitely many solutions. We also find out all the existing solutions.

1. INTRODUCTION. Let $D_1, D_2$and λ denote positive integers such that $\gcd(D_1, D_2) = 1$. Then the equation

$D_1x^2 + D_2 = \lambda y^n$ in positive integers in x, y and $n \geq 2$,

has a rich history and it has attracted the attention of several great mathematicians. Several papers have been written on these topics, specially for particular values of D_1, D_2 and λ. Detailed surveys on this equation can be found in [1] . In this note we prove the above equation which is a particular case. We use Mihailescu's theorem [2] (Catalan's conjecture) in our proof.

2. Lemma : We start with some lemma which is required in our proof.

Lemma. *2.1 : If r and p are primes such that $\gcd(r-1, p) = 1$, then $y^p \equiv x$ (mod r) $\implies$ $y \equiv x$(mod r) for $0 \leq x \leq r-1$.*

Proof. If $x = 0$ then it is obvious.

Let $1 \leq x \leq r-1$, $x^{\Phi(r)} \equiv 1$ (*mod r*)

$x^{r-1} \equiv 1$ (*mod r*) .

$\gcd(r-1, p) = 1 \implies \exists s, d \in \mathbb{Z}$ such that $(r-1)s + pd = 1$.

$x = x^1 = x^{(r-1)s+pd} = x^{(r-1)s}x^{pd} = x^{pd}$.

Let us assume that $x^p = z^p$ for some $1 \leq z \leq r-1$.

Similarly $z = z^{pd}$.

$x = x^{pd} = (x^p)^d = (z^p)^d = z^{pd} = z$ i.e. $x = z$ and we are done. □

Corollary. *2.2 If r and p are primes such that $r \leq p$, then $y^p = x$ (mod r) $\implies y \equiv x$ (mod r).*

Theorem. *3.1 The equation $2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} + 1 = y^2$ $\alpha_1, \alpha_2, \alpha_3, y \in \mathbb{N}$has only five solutions and they are*

$(\alpha_1, \alpha_2, \alpha_3, y) =$

(1) $(3, 2, 1, 19)$

(2) $(3, 1, 1, 11)$

(3) $(5, 1, 2, 49)$

(4) $(6, 4, 1, 161)$

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(5) $(6, 1, 1, 31)$

Proof. Consider the diophantine equation
$2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} + 1 = y^2$ ...... (1), $\alpha_1, \alpha_2, \alpha_3, y \in \mathbb{N}$
$2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} = (y+1)(y-1)$
$2|^{\alpha_1-2}3^{\alpha_2}5^{\alpha_3} = k(k+1)$, $k \in \mathbb{N}$
There will be six different cases ...
(1) $k = 2^{\alpha_1-2}$, $k+1 = 3^{\alpha_2}5^{\alpha_3}$
(2) $k = 3^{\alpha_2}$, $k+1 = 2^{\alpha_1-2}5^{\alpha_3}$
(3) $k = 5^{\alpha_3}$, $k+1 = 2^{\alpha_1-2}3^{\alpha_2}$
(4) $k = 2^{\alpha_1-2}3^{\alpha_2}$, $k+1 = 5^{\alpha_3}$
(5) $k = 2^{\alpha_1-2}5^{\alpha_3}$, $k+1 = 3^{\alpha_2}$
(6) $k = 3^{\alpha_2}5^{\alpha_3}$, $k+1 = 2^{\alpha_1-2}$
CASE 1 - $k+1 = 3^{\alpha_2}5^{\alpha_3}$, $k = 2^{\alpha_1-2}$
$3^{\alpha_2}5^{\alpha_3} - 2^{\alpha_1-2} = 1$
$3^{\alpha_2}5^{\alpha_3} - 2^r = 1$, $r = \alpha_1 - 2 \in \mathbb{N}$
Now $2^r \equiv -1$ (*mod* 5) $\Longrightarrow r \equiv 2$ (*mod* 4)
$2^r \equiv -1$ (*mod* 3) $\Longrightarrow r \equiv 1$ (*mod* 2) , which is a contradiction. Therefore case 1 has no solution.
CASE 2 - $K = 3^{\alpha_2}$, $K+1 = 2^{\alpha_1-2}5^{\alpha_3}$
$2^{\alpha_1-2}5^{\alpha_3} - 1 = 3^{\alpha_2}$
$2^{\alpha_1-2}5^{\alpha_3} = 1 + 3^{\alpha_2}$
$3^{\alpha_2} \equiv -1$ (*mod* 5) $\Longrightarrow \alpha_2 \equiv 2$ (*mod* 4)
So $\alpha_2 = 2, 6, 10, ...$
put $\alpha_2 = 2$ we get $\alpha_1 = 3$, $\alpha_3 = 1$ i.e. $2^3 3^2 5^1 + 1 = 19^2$.
Hence $(3, 2, 1, 19)$ is a solution of the equation.
Next, we prove that $\alpha_2 = 2^t$ for some $t \in \mathbb{N}$.
If not , let α_2 has an odd prime factor $p \geq 3$.
$2^{\alpha_1-2}5^{\alpha_3} = 1 + z^p$, $z \in \mathbb{N}$.
$2^{\alpha_1-2}5^{\alpha_3} = (1+z)(z^{p-1} - z^{p-2} + ... + 1) = AB$, where $A = 1 + z$, $B = z^{p-1} - z^{p-2} + ... + 1$
$\gcd(A, B) = 1$ or p.
Now $z^p \equiv -1$ (*mod* 2) $\Longrightarrow z \equiv -1$ (*mod* 2) since $\gcd(1, p) = 1$.
$z^p \equiv -1$ (*mod* 5) $\Longrightarrow z \equiv -1$ (*mod* 5) since $\gcd(4, p) = 1$.
Therefore $2 \mid A$, $5 \mid A$ and $\gcd(A, B) = 1$ or p .
So the equation has no solution whenever α_2 has an odd prime factor i.e. $\alpha_2 = 2$ is the only possible solution.
CASE 3 - $k = 5^{\alpha_3}$, $k+1 = 2^{\alpha_1-2}3^{\alpha_2}$
$2^{\alpha_1-2}3^{\alpha_2} = 1 + 5^{\alpha_3}$.
$5^{\alpha_3} \equiv -1$ (*mod* 3) $\Longrightarrow \alpha_3 \equiv 1$ (*mod* 2) .
put $\alpha_3 = 1$ we get $\alpha_1 = 3$, $\alpha_2 = 1$ i.e. $2^3 3^1 5^1 + 1 = 121 = 11^2$.
Therefore $(3, 1, 1, 11)$ is a solution.
For $\alpha_3 \geq 2$, $\alpha_3$has at least one odd prime factor $p \geq 3$.
Therefore $2^{\alpha_1-2}3^{\alpha_2} = 1 + z^p$ for some $z \in \mathbb{N}$.
$2^{\alpha_1-2}3^{\alpha_2} = AB$, where $A = 1 + z$, $B = z^{p-1} - z^{p-2} + ... + 1$.

$\gcd(A,B) = 1$or p .
$z^p \equiv -1$ (mod 2) $\implies z \equiv -1$ (mod 2) since $\gcd(1,p) = 1$.
$z^p \equiv -1$ (mod 3) $\implies z \equiv -1$ (mod 3) since $\gcd(2,p) = 1$.
Therefore $2 \mid A$, $3 \mid A$ and the equation has no more solution in this case.
CASE 4 - $k+1 = 5^{\alpha_3}$, $k = 2^{\alpha_1-2}3^{\alpha_2}$
$2^{\alpha_1-2}3^{\alpha_2} = 5^{\alpha_3} - 1$
$5^{\alpha_3} \equiv 1$ (mod 3) $\implies \alpha_3 \equiv 0$ (mod 2)
Therefore $2^{\alpha_1-2}3^{\alpha_2} = z^2 - 1$ for some $z \in \mathbb{N}$.
$2^{\alpha_1-4}3^{\alpha_2} = t(t+1)$, $t \in \mathbb{N}$.
$2^r3^{\alpha_2} = t(t+1)$, $r = \alpha_1 - 4 \in \mathbb{N}$.
Now by Mihailescu's theorem [2] (Catalans's conjecture) 8 and 9 are only consecutive perfect powers.
Therefore $2^r - 3^{\alpha_2} = \pm 1$ has only one solution $r = 3$, $\alpha_2 = 2$ for $r \geq 2$ and $\alpha_2 \geq 2$.
But $r = 3$, $\alpha_2 = 2$ i.e. $\alpha_1 = 7$,$\alpha_2 = 2$ is not a solution of the equation.
Now we consider the cases $r = 1$ and $\alpha_2 = 1$.
If $\alpha_2 = 1$ then either $r = 2$ or $r = 1$.
If $\alpha_2 = 1$ and $r = 2$ i.e. $\alpha_1 = 6$, $\alpha_2 = 1$ which is not a solution of the equation.
If $\alpha_2 = 1$ and $r = 1$ i.e. $\alpha_1 = 5$, $\alpha_2 = 1$ and $\alpha_3 = 2$.
Therefore we have a solution $2^5 3^1 5^2 + 1 = 49^2$ i.e. $(5, 1, 2, 49)$.
CASE 5- $k+1 = 3^{\alpha_2}$, $k = 2^{\alpha_1-2}5^{\alpha_3}$.
$2^{\alpha_1-2}5^{\alpha_3} = 3^{\alpha_2} - 1$.
$3^{\alpha_2} \equiv 1$ (mod 5) $\implies \alpha_2 \equiv 0$ (mod 4)
Therefore $2^{\alpha_1-2}5^{\alpha_3} = z^2 - 1$ for some $z \in \mathbb{N}$.
$2^{\alpha_1-4}5^{\alpha_3} = t(t+1)$ for some $t \in \mathbb{N}$ i.e. $2^r5^{\alpha_3} = t(t+1)$; $r = \alpha_1 - 4 \in \mathbb{N}$.
Now by Mihailescu's theorem [2], $2^r - 5^{\alpha_3} = \pm 1$ has no solution for $r \geq 2$ and $\alpha_3 \geq 2$.
Now we consider the cases $r = 1$ and $\alpha_3 = 1$.
If $r = 1$ then the equation has no solution.
If $\alpha_3 = 1$ then $r = 2$ i.e. $\alpha_1 = 6$, then $\alpha_2 = 4$.
Therefore $2^6 3^4 5^1 + 1 = 161^2$i.e. $(6, 4, 1, 161)$ is a solution of the equation.
CASE 6 - $k+1 = 2^{\alpha_1-2}$, $k = 3^{\alpha_2}5^{\alpha_3}$.
$2^{\alpha_1-2} - 3^{\alpha_2}5^{\alpha_3} = 1$
$3^{\alpha_2}5^{\alpha_3} = 2^r - 1$, $r = \alpha_1 - 2 \in \mathbb{N}$.
$2^r \equiv 1$ (mod 5) $\implies r \equiv 0$ (mod 4)
Therefore $3^{\alpha_2}5^{\alpha_3} = z^2 - 1$, $z = 2^\beta$ for some $\beta \in \mathbb{N}$.
$3^{\alpha_2}5^{\alpha_3} = (z+1)(z-1)$
Either $3^{\alpha_2} = z + 1$ and $5^{\alpha_3} = z - 1$ or $3^{\alpha_2} = z - 1$ and $5^{\alpha_3} = z + 1$.
If $5^{\alpha_3} = z - 1$ i.e. $5^{\alpha_3} = 2^\beta - 1$
$5^{\alpha_3} - 2^\beta = -1$ has no solution for $\alpha_3 \geq 2$ and $\beta \geq 2$ by Mihailescu's theorem [2].
It has also no solution for $\alpha_3 = 1$ and $\beta = 1$.
If $5^{\alpha_3} = z + 1$ i.e. $5^{\alpha_3} = 2^\beta + 1$

$5^{\alpha_3} - 2^{\beta} = 1$ has no solution for $\alpha_3 \geq 2$ and $\beta \geq 2$ by Mihailescu's theorem [2].

If $\beta = 1$ then it has no solution in this case.

If $\alpha_3 = 1$ then $\beta = 2 \implies \alpha_2 = 1$ and $\alpha_1 = 6$.

Therefore $2^6 3^1 5^1 + 1 = 31^2$ i.e.$(6, 1, 1, 31)$ is a solution. □

Theorem. *3.2 The equation* $2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} + 1 = y^p, \alpha_1, \alpha_2, \alpha_3, y \in \mathbb{N}$ *and* $p \geq 3$ *be a prime ; has no solution.*

Proof. Let $2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} + 1 = y^p$, $p \geq 3$ be a prime.

$2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} = AB$ where $A = 1 + y$ and $B = y^{p-1} - y^{p-2} + ... + 1$.

$\gcd(A, B) = 1$ or p.

Now $y^p \equiv 1 \ (mod\ p_i)$,$p_i = 2, 3, 5$.

$\implies y \equiv 1 \ (\ mod\ p_i\ )$ since $\gcd(p_i - 1, p) = 1$.

Hence the equation has no solution. □

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