# Lyapunov-type inequality for the Hadamard fractional boundary value problem on a general interval $[a;b], (1 \leqslant a < b)$

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**Abstract:** In this paper, we studied an open problem, where using two different methods, we obtained several results for a Lyapunov-type and Hartman-Wintner-type inequalities for a Hadamard fractional differential equation on a general interval $[a;b], (1 \leqslant a < b)$ with the boundary value conditions.

**Keywords:** Hadamard fractional derivative; boundary value problem; Green's function; Lyapunov-type inequality; Hartman-Wintner-type inequality.

## 1 introduction

The first result in this domain is due to Lyaponov [1], can be stated as follows: If a nontrivial continuous solution to the following boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0 \end{cases} \tag{1}$$

existe, where $q : [a;b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{2}$$

Recently, several articles from the inequality of Lyapunov have been published about a differential equations of the integer order and fractional order, see [5-10] and references therein, for example: The following result for the Riemann-Liouville fractional boundary value problem is found by D. O'Regan and B. Samet [4]

$$\begin{cases} {}^{R}D^{\alpha}u(t) + q(t)u(t) = 0, & a < t < b, \quad 3 < \alpha \leqslant 4, \\ u(a) = u'(a) = u''(a) = u''(b) = 0 \end{cases} \tag{3}$$

has a nontrivial continuous solution, then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}. \tag{4}$$

In [2] Qinghua, Chao and Jinxun established a Lyapunov-type inequality for a differential equation that depands on the Hadamard fractional derivative, for the boundary value problem

$$\begin{cases} {}^H D^{\alpha} u(t) - q(t)u(t) = 0, \qquad 1 < t < e, \quad 1 < \alpha \leqslant 2, \\ u(1) = u(e) = 0 \end{cases} \tag{5}$$

where $q : [1; e] \to \mathbb{R}$ is a continuous function.They proved that if a nontrivial continuous solution to the above problem, then

$$\int_1^e |q(s)|ds > \Gamma(\alpha)\lambda^{1-\alpha}(1-\lambda)^{1-\alpha} \exp \lambda, \tag{6}$$

where $\lambda = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$.
And they have presented the following open problem for readers:
How to get the Lyapunov inequality for the following the Hadarmard fractional value problem (HFBVP)

$$\begin{cases} {}^H D^{\alpha} u(t) - q(t)u(t) = 0, \qquad 1 \leqslant a < t < b, \quad 1 < \alpha \leqslant 2, \\ u(a) = u(b) = 0 \end{cases} \tag{7}$$

where ${}^H D^{\alpha}$ is the Hadamard fractional derivative, and $q : [a; b] \to \mathbb{R}$ is a continuous function.
In this paper we answered the previous question by using two methods, and also we get the Hartman-Wintner-type inequalities.

# 2 Preliminaries

**Definition 1** *[3] Let $a, b, \alpha \in \mathbb{R}^+$ where $a < b$ and $n-1 < \alpha \leqslant n$ with $n \in \mathbb{N}^*$, The Hadamard fractional integral of ordre $\alpha$.for a function $f \in L^1[a,b]$ is defined as*

$${}_a^H I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \;\; a < t < b \tag{8}$$

*with $\Gamma$ is Gamma Euler function*

**Definition 2** *[3] Let $a, b \in \mathbb{R}^+$ with $a < b$, The Hadamard fractional derivative of ordre $\alpha \in \mathbb{R}^+$.for a function $f \in L^1[a,b]$ is defined as*

$${}_a^H D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} t^n \frac{d^n}{dt^n} \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}, \;\; a < t < b \tag{9}$$

*where $n-1 < \alpha \leqslant n$ with $n \in \mathbb{N}^*$*

**Lemma 3** *[3] Let $0 \leqslant a < b$ and $\alpha > 0$ where $n-1 < \alpha \leqslant n$. and $n \in \mathbb{N}^*$ The equation ${}^{H}D^{\alpha}u(t) = 0$ has as its solutions*

$$u(t) = \sum_{i=1}^{i=n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, \quad t \in [a,b] \tag{10}$$

*and moreover*

$${}^{H}I^{\alpha}\ {}^{H}D^{\alpha}u(t) = u(t) + \sum_{i=1}^{i=n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, \tag{11}$$

*where $c_i \in \mathbb{R}$, $(i = 1, ..., n)$ are constants.*

**Lemma 4** *Let $A, B \in \mathbb{R}$, we have*

$$AB \leqslant \frac{(A+B)^2}{4} \tag{12}$$

# 3 Main results

**Lemma 5** *Let $u \in C([a;b], \mathbb{R})$, the following problem*

$$\begin{cases} {}^{H}D^{\alpha}u(t) - q(t)u\,(t) = 0, \qquad 1 \leqslant a < t < b, \quad 1 < \alpha \leqslant 2, \\ \\ u(a) = u\,(b) = 0 \end{cases} \tag{13}$$

*has equivalent to the fractional integral equation*

$$u\,(t) = \int_a^b G\,(t,s)\,q(s)u\,(s)\,ds \tag{14}$$

*where*

$$G\,(t,s) = \begin{cases} g_1(t,s) = g_2(t,s) + \frac{1}{\Gamma(\alpha)}\left(\ln \frac{t}{s}\right)^{\alpha-1}\frac{1}{s}, \qquad a \leqslant s \leqslant t \leqslant b, \\ \\ g_2(t,s) = -\frac{1}{\Gamma(\alpha)}\frac{\left(\ln \frac{t}{a}\right)^{\alpha-1}\left(\ln \frac{b}{s}\right)^{\alpha-1}}{\left(\ln \frac{b}{a}\right)^{\alpha-1}}\frac{1}{s}, \qquad a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{15}$$

*with $1 \leqslant a < b$.*

**Proof.** Using Lemma 3, we have

$$u\,(t) = c_1\left(\ln \frac{t}{a}\right)^{\alpha-1} + c_2\left(\ln \frac{t}{a}\right)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s)u\,(s)\,\frac{ds}{s} \tag{16}$$

where $c_1, c_2 \in \mathbb{R}$
using the boundary condition $u(a) = u(b) = 0$ we get $c_2 = 0$ and

$$c_1 = -\frac{\left(\ln \frac{b}{a}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} q(s)u\,(s)\,\frac{ds}{s} \tag{17}$$

Substituting the values of $c_1$ and $c_2$ in (16), we obtain

$$\begin{aligned}
u(t) &= -\frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1}q(s)u(s)\frac{ds}{s} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}q(s)u(s)\frac{ds}{s} \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t\left[\left(\ln\frac{t}{s}\right)^{\alpha-1}-\left(\ln\frac{b}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}\right]q(s)u(s)\frac{ds}{s} \\
&\quad-\frac{1}{\Gamma(\alpha)}\int_t^b\left(\ln\frac{b}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}q(s)u(s)\frac{ds}{s} \\
&= \int_a^b G(t,s)\,q(s)u(s)\,ds \qquad (18)
\end{aligned}$$

the proof is complete ■

**Lemma 6** *The Green's function $G$ defined in Lemma 5, has the following properties*
*i) $G(t,s)\leqslant g_2(s,s)\leqslant 0$, for all $(t,s)\in[a,b]\times[a,b]$*
*ii) For any $s\in[a,b]$*

$$|G(t,s)|\leqslant|G(s,s)|=-g_2(s,s)\leqslant\frac{1}{4^{(\alpha-1)}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1} \qquad (19)$$

***Proof.*** *We start by fixing an arbitrary $s\in[a,b]$. Differentiating $G(t,s)$ with respect to $t$, we get*
*For $1\leqslant a\leqslant t\leqslant s\leqslant b$, we have*

$$\frac{\partial}{\partial t}g_2=-\frac{(\alpha-1)}{\Gamma(\alpha)st}\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\leqslant 0, \qquad (20)$$

*we obtain*

$$g_2(s,s)\leqslant g_2(t,s)\leqslant g_2(a,s)=0, \qquad (21)$$

*while for $1\leqslant a\leqslant s\leqslant t\leqslant b$, we have*

$$\begin{aligned}
\frac{\partial}{\partial t}g_1 &= \frac{\partial}{\partial t}g_2+\frac{(\alpha-1)}{\Gamma(\alpha)st}\left(\ln\frac{t}{s}\right)^{\alpha-2} \\
&= -\frac{(\alpha-1)}{\Gamma(\alpha)st}\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}+\frac{(\alpha-1)}{\Gamma(\alpha)st}\left(\ln\frac{t}{s}\right)^{\alpha-2} \\
&= \frac{(\alpha-1)}{\Gamma(\alpha)st}\left[\left(\ln\frac{t}{s}\right)^{\alpha-2}-\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\right]
\end{aligned}$$

$$\begin{aligned} &= \frac{(\alpha-1)\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha)st}\left[\frac{\left(\ln\frac{t}{s}\right)^{\alpha-2}}{\left(\ln\frac{t}{a}\right)^{\alpha-2}} - \frac{\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\right] \\ &= \frac{(\alpha-1)\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha)st}\left[\left(\frac{\ln\frac{t}{a}}{\ln\frac{t}{s}}\right)^{2-\alpha} - \left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1}\right] \end{aligned} \tag{22}$$

*by* $1 \leqslant a \leqslant s \leqslant t \leqslant b$ *we get*

$$\left(\frac{\ln\frac{t}{a}}{\ln\frac{t}{s}}\right)^{2-\alpha} \geqslant 1, \tag{23}$$

*and*

$$-\left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1} \geqslant -1 \tag{24}$$

*using* (23) *and* (24) *we obtain*

$$\left[\left(\frac{\ln\frac{t}{s}}{\ln\frac{t}{a}}\right)^{\alpha-2} - \left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1}\right] \geqslant 0 \tag{25}$$

*So thus*

$$\frac{\partial}{\partial t}g_1 \geqslant 0 \tag{26}$$

*Using* $1 \leqslant a \leqslant s \leqslant t \leqslant b$ *we get*

$$g_1(s,s) \leqslant g_1(t,s) \leqslant g_1(b,s) = 0 \tag{27}$$

*We obtain*

$$g_2(s,s) \leqslant g_2(t,s) \leqslant g_1(t,s) \leqslant 0, \tag{28}$$

*hence*

$$G(t,s) \leqslant 0 \tag{29}$$

*We prove that*

$$|G(s,s)| \leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1} \tag{30}$$

*we have* $G(s,s) = g_2(s,s) = g_1(s,s) \leqslant g_2(t,s) \leqslant g_1(t,s) \leqslant 0.$
*Using Lemma 4, we have*

$$\begin{aligned} |G(s,s)| &= \frac{1}{\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}s}\left[\left(\ln\frac{s}{a}\right)\left(\ln\frac{b}{s}\right)\right]^{\alpha-1} \\ &\leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}s}\left[\left(\ln\frac{s}{a}+\ln\frac{b}{s}\right)^2\right]^{\alpha-1} \\ &= \frac{1}{4^{\alpha-1}\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}s}\left[\left(\ln\frac{b}{a}\right)^2\right]^{\alpha-1} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4^{\alpha-1}\Gamma(\alpha)s}\left(\ln\frac{b}{a}\right)^{\alpha-1} \\
&\leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1}
\end{aligned}
$$

*Therefore*

$$
|G(t,s)| \leqslant |G(s,s)| = -g_2(s,s) \leqslant \frac{1}{4^{(\alpha-1)}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1} \tag{31}
$$

*The proof is complete* ■

We have the following Hartman-Wintner-type inequality.

**Theorem 7** *If a nontrivial continuous solution to the Hadamard fractional boundary value problem* (7) *existe, then*

$$
\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\,ds \geqslant \left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha) \tag{32}
$$

**Proof.** Let $E = C\left([a,b],\mathbb{R}\right)$ be the Banach space endowed with the norm

$$
\|u\| = \sup_{t\in[a,b]}|u(t)|
$$

we have

$$
|u(t)| \leqslant \int_a^b |G\left(t,s\right)|\,|q(s)|\,|u\left(s\right)|\,ds
$$

which yields

$$
\|u\| \leqslant \|u\| \int_a^b |g_2\left(s,s\right)|\,|q(s)|\,|u\left(s\right)|\,ds
$$

Since $u$ is non trivial, then $\|u\| \neq 0$, so

$$
1 \leqslant \int_a^b \frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha)s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\,ds
$$

from which the inequality in (32) follows ■

**Corollary 8** *If a nontrivial continuous solution to the Hadamard fractional boundary value problem existe, then*

$$
\int_a^b \left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\,ds \geqslant a\left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha) \tag{33}
$$

**Proof.** from theorem 7, we have

$$\int_a^b \frac{1}{s}\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\,ds \geqslant \left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha)$$

nexte we not $\frac{1}{a} \geqslant \frac{1}{s}$
thus we get

$$\int_a^b \left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}|q(s)|\,ds \geqslant a\left(\ln\frac{b}{a}\right)^{\alpha-1}\Gamma(\alpha) \tag{34}$$

■

We have the following Lyapunov-type inequality.

**Theorem 9** *If a nontrivial continuous solution to the Hadamard fractional boundary value problem* (7) *existe, then*

$$\int_a^b |q(s)|\,ds \geqslant 4^{(\alpha-1)}\Gamma(\alpha)a\left(\ln\frac{b}{a}\right)^{1-\alpha} \tag{35}$$

**Proof.** from the corollary 8, we have

$$\int_a^b |q(s)|\,ds \geqslant a\left(\ln\frac{b}{a}\right)^{\alpha-1}\frac{\Gamma(\alpha)}{\max\limits_{s\in[a,b]} h(s)} \tag{36}$$

where

$$h(s) = \left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1} \tag{37}$$

If $s = a$ or $s = b$ then $h(s) = 0$
Else if $s \in ]a, b[$ we differentiate $h(s)$

$$\begin{aligned} h'(s) &= \frac{(\alpha-1)}{s\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{2-\alpha}}\left(\ln\frac{b}{s}-\ln\frac{s}{a}\right) \\ &= \frac{(\alpha-1)\left(\ln\frac{ab}{s^2}\right)}{s\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{2-\alpha}} \end{aligned}$$

we have only one solution

$$s_0 = \sqrt{ab} \tag{38}$$

of the equation $h'(s) = 0$ on $]a; b[$. We obtain

$$\max_{s\in[a,b]} h(s) = h(s_0) = \left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)^{\alpha-1} \tag{39}$$

We have

$$ab = \sqrt{ab}\sqrt{ab} \Leftrightarrow \ln\frac{\sqrt{ab}}{a} = \ln\frac{b}{\sqrt{ab}} \Leftrightarrow \left(\ln\frac{\sqrt{ab}}{a}-\ln\frac{b}{\sqrt{ab}}\right)^2 = 0$$

$$\Leftrightarrow \quad 4\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)=\left(\ln\frac{\sqrt{ab}}{a}\right)^2+\left(\ln\frac{b}{\sqrt{ab}}\right)^2+2\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)$$

$$\Leftrightarrow \quad \left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)=\frac{1}{4}\left(\ln\frac{\sqrt{ab}}{a}+\ln\frac{b}{\sqrt{ab}}\right)^2=\frac{1}{4}\left(\ln\frac{b}{a}\right)^2$$

$$\Leftrightarrow \quad \left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)^{\alpha-1}=\frac{1}{4^{(\alpha-1)}}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)} \tag{40}$$

by (39) and (40)

$$\max_{s\in[a,b]}h(s)=h(s_0)=\frac{1}{4^{(\alpha-1)}}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)} \tag{41}$$

we substiting (41) into (36) we obtain

$$\int_a^b|q(s)|\,ds\geqslant 4^{(\alpha-1)}\Gamma(\alpha)a\left(\ln\frac{b}{a}\right)^{1-\alpha}$$

The proof is complete ■

We define the constants:

$$\xi_1=\exp\left(\frac{1}{2}\left[\left[2\left(\alpha-1\right)+\ln ba\right]-\sqrt{4\left(\alpha-1\right)^2+\ln^2\frac{b}{a}}\right]\right) \tag{42}$$

and

$$\xi_2=\exp\left(\frac{1}{2}\left[\left[2\left(\alpha-1\right)+\ln ba\right]+\sqrt{4\left(\alpha-1\right)^2+\ln^2\frac{b}{a}}\right]\right). \tag{43}$$

**Lemma 10** *The function $G$ defined in Lemma 5, satisfie the following property*

$$\max_{t,s\in[a,b]}|G(t,s)|=\frac{1}{\Gamma(\alpha)\xi_1}\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{\alpha-1}. \tag{44}$$

**Proof.** we have $\max_{t,s\in[a,b]}|G(t,s)|=\max_{s\in[a,b]}|g_2(s,s)|$

where

$$g_2(s,s)=-\frac{1}{\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}}\frac{\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}}{s}$$

It follows that we only need to get the maximum value of the function

$$f(s)=\frac{\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}}{s} \tag{45}$$

we observe that $f(a) = f(b) = 0$.
If $s \in ]a, b[$, differentiate $f(s)$

$$f'(s) = \left[ (\alpha - 1) \frac{\ln \frac{b}{s} - \ln \frac{s}{a}}{\left(\ln \frac{s}{a} \ln \frac{b}{s}\right)} - 1 \right] \left( \ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha - 1} \frac{1}{s^2}$$

we have

$$\begin{aligned} f'(s) = 0 \quad & \Leftrightarrow (\alpha - 1) \left(\ln \tfrac{b}{s} - \ln \tfrac{s}{a}\right) = \ln \tfrac{s}{a} \ln \tfrac{b}{s} \\ & \Leftrightarrow [2(\alpha - 1) + \ln b + \ln a] \ln s - [(\alpha - 1) + \ln b] \ln a - \ln^2 s - (\alpha - 1) \ln b = 0 \\ & \Leftrightarrow \ln^2 s - [2(\alpha - 1) + \ln ba] \ln s + [(\alpha - 1) \ln ba + \ln b \ln a] = 0 \\ & \Leftrightarrow x^2 - [2(\alpha - 1) + \ln ba] x + [(\alpha - 1) \ln ba + \ln b \ln a] = 0 \end{aligned}$$

where $x = \ln s$.
we get

$$\begin{cases} x_1 = \frac{[2(\alpha-1) + \ln ba] - \sqrt{\Delta}}{2} = \ln \xi_1 \\ \\ x_2 = \frac{[2(\alpha-1) + \ln ba] + \sqrt{\Delta}}{2} = \ln \xi_2 \end{cases} \tag{46}$$

where

$$\Delta = 4(\alpha - 1)^2 + \ln^2 \frac{b}{a} \tag{47}$$

we have

$$\begin{aligned} x_2 \quad & > \quad \frac{\ln ba + \sqrt{\left(\ln \frac{b}{a}\right)^2}}{2} = \ln b \\ & \Rightarrow \quad \xi_2 \notin ]a; b[ \end{aligned}$$

Also we have

$$\begin{aligned} x_1 \quad & = \quad \frac{1}{2} \left( 2(\alpha - 1) + \ln ba - \sqrt{\left(2(\alpha - 1) + \ln \frac{b}{a}\right)^2 - 4(\alpha - 1)\left(\ln \frac{b}{a}\right)} \right) \\ & > \quad \frac{1}{2} \left( 2(\alpha - 1) + \ln ba - \sqrt{\left(2(\alpha - 1) + \ln \frac{b}{a}\right)^2} \right) \\ & = \quad \frac{1}{2} \left( 2(\alpha - 1) + \ln ba - 2(\alpha - 1) - \ln \frac{b}{a} \right) = \ln a \\ & \Rightarrow \quad \xi_1 > a \end{aligned}$$

and

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(2(\alpha-1)+\ln ba-\sqrt{\left(2(\alpha-1)-\ln\frac{b}{a}\right)^2+4(\alpha-1)\ln\frac{b}{a}}\right)\\
&< \frac{1}{2}\left(2(\alpha-1)+\ln ba-\sqrt{\left(2(\alpha-1)-\ln\frac{b}{a}\right)^2}\right)\\
&= \frac{1}{2}\left(2(\alpha-1)+\ln ba-\left|2(\alpha-1)-\ln\frac{b}{a}\right|\right)\\
&\leqslant \frac{1}{2}\left(2(\alpha-1)+\ln ba-\left(|2(\alpha-1)|-\left|\ln\frac{b}{a}\right|\right)\right)\\
&= \frac{1}{2}\left(2(\alpha-1)+\ln ba-\left(2(\alpha-1)-\ln\frac{b}{a}\right)\right)=\ln b\\
&\Rightarrow\ \xi_1<b
\end{aligned}$$

we obtient $\xi_1 \in ]a;b[$

Hence

$$\max_{s\in[a,b]}|f(s)|=\frac{1}{\xi_1}\left(\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}\right)^{\alpha-1} \tag{48}$$

Therefore

$$\max_{t,s\in[a,b]}|G(t,s)|=\frac{1}{\Gamma(\alpha)\xi_1}\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{\alpha-1}. \tag{49}$$

The proof is complete ■

We have the following Lyapunov-type inequality.

**Theorem 11** *If a nontrivial continuous solution to the HFBVP* (7) *existe, then*

$$\int_a^b|q(s)|\,ds\geqslant\Gamma(\alpha)\xi_1\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{1-\alpha}, \tag{50}$$

*where*

$$\xi_1=\exp\left(\frac{1}{2}\left[\left[2(\alpha-1)+\ln ba\right]-\sqrt{4(\alpha-1)^2+\ln^2\frac{b}{a}}\right]\right).$$

**Proof.** By Lemma 5, the solution of the HFBVP can be written as

$$u(t)=\int_a^b G(t,s)\,q(s)u(s)\,ds$$

Thus for all $t \in [a,b]$ we have

$$\begin{aligned} |u(t)| &\leqslant \int_a^b |G(t,s)|\,|q(s)|\,|u(s)|\,ds \\ &\leqslant \|u\| \int_a^b |G(t,s)|\,|q(s)|\,ds \end{aligned}$$

which yields

$$\|u\| \leqslant \|u\| \int_a^b |G(t,s)|\,|q(s)|\,ds$$

Since $u$ is non trivial, then $\|u\| \neq 0$, so

$$1 \leqslant \int_a^b |G(t,s)|\,|q(s)|\,ds$$

New, an application of Lemma 10, we obtain

$$\int_a^b |q(s)|\,ds \geqslant \Gamma(\alpha)\xi_1 \left( \frac{\ln \frac{\xi_1}{a} \ln \frac{b}{\xi_1}}{\ln \frac{b}{a}} \right)^{1-\alpha}$$

The proof is complete ■

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