The Corrections of Pressure And Mass of The White Dwarf Star

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Abstract Traditional explanation for the white dwarf star is based on the ideally degenerate Fermi electron gas to produce the pressure against gravity. This theory predicts the upper mass limit of the white dwarf star is 1.44 times as large as our sun although the Fermi electron gas is calculated at the temperature T of absolute zero. In this research, first considering the electron-electron interaction in the high-density Fermi electron gas at T=0 K, this interaction causes the pressure 2/137 time less than the original value. Then we estimate the temperature effect using statistical mechanics and find the total pressure is mainly proportional to $T/(\ln T)$ at the given particle number N and volume V. The Fermi electron gas is about 4.343 times at 10^8 K larger than it at 10^7 K. According to this, the relationship between radius and mass of the white dwarf star is obtained and it depends on temperature, electron mass, and proton mass. An unknown parameter δ=0.02716 is calculated by using an example of a white dwarf star with the mass of our sun and the radius of Earth. This relationship is useful for estimating the inner temperature of a white dwarf star.

Keywords: white dwarf star, degenerate Fermi electron gas, pressure, upper mass limit, electron-electron interaction

I. Introduction

The white dwarf star has been investigated many years and it was named first in 1922 [1]. It is thought to be the type of the low to medium mass stars in the final evolution stage. The white dwarf star usually has very high density with the mass similar to our sun but the volume small like Earth. The reported largest mass seems to be the one found in 2007 which is 1.33 times as large as our sun [2]. The early theory to explain its mass upper limit is based on the ideally degenerate Fermi electron gas [3-7]. The calculation adopts all electrons like free particles occupying all energy levels until to Fermi energy as they are at zero temperature. It is surprising that even in the high-temperature and high-pressure situation, the ideal Fermi gas still works. It makes the curiosity to discuss the temperature effect by statistical mechanics.

Since Einstein proposed General Relativity in 1915, some appropriate metrics have been found such as the Schwarzschild metric, the Kerr metric, and the Kerr-Newman metric [8-11]. Especially, the Kerr-Newman metric describes the rotating and charged star. Some detail problems about the Kerr-Newman black hole have been discussed [12,13]. As we know, most stars are rotating and they might be also easily charged because the relativistically massive particles escaping the gravity. According to statistical mechanics, the relativistic electrons have more possibility to escape gravity than helium nuclei at the same high temperature. Because of this factor, we consider the positive charged star and the Coulomb interaction existing in the rest positive charges, and further calculate the pressure produced by these rest charges. The Coulomb force also an important one against gravity so the upper mass limit of the white dwarf star should be higher.

II. The Degenerate Fermi Electron Gas For The White Dwarf Star

First, we review the calculation of the upper mass limit for the dwarf star. It adopt the ideally degenerate Fermi electron gas and considers the relativistic kinetic energy in the calculation [3,4]. Because the electron has spin s= $\pm\frac{1}{2}$, each energy state permits two electrons occupied. Each electron has the rest mass m_e, and its relativistic kinetic energy E at momentum p is

$$E_k = m_e c^2 \left\{\left[1 + \left(\frac{p}{m_e c}\right)^2\right]^{1/2} - 1\right\}. \tag{1}$$

The Fermi electron gas with the total number N and total volume V has total kinetic energy

$$\begin{aligned} E_0 &= 2m_e c^2 \sum_{|\vec{p}|<p_F} \left\{\left[1 + \left(\frac{\vec{p}}{m_e c}\right)^2\right]^{1/2} - 1\right\} \\ &= \frac{2Vm_e c^2}{h^3}\int_0^{p_F} dp 4\pi p^2 \left\{\left[1 + \left(\frac{\vec{p}}{m_e c}\right)^2\right]^{1/2} - 1\right\}, \end{aligned} \tag{2}$$

where h is the Planck's constant and p_F is Fermi momentum defined as

$$p_F = h\left(\frac{3N}{8\pi V}\right)^{1/3}. \tag{3}$$

Considering the mass m_p of a proton and the mass m_n of a neutron, the total mass M of a white dwarf star mainly consisting of helium nuclei is

$$M = \left(m_e + m_p + m_n\right)N \approx 2m_p N \approx 2m_n N. \tag{4}$$

If we define the parameter

$$x_F \equiv \frac{p_F}{m_e c} = \frac{h}{2m_e c}\left(\frac{3N}{8\pi V}\right)^{1/3}, \tag{5}$$

then Eq. (1) becomes

$$E_0 = \frac{8\pi m_e^4 c^5 V}{h^3}\left[f(x_F) - \frac{1}{3}x_F^3\right], \tag{6}$$

where

$$f(x_F) = \int_0^{x_F} dx x^2 \left[(1 + x^2)^{1/2}\right]. \tag{7}$$

The pressure produced by the ideal Fermi electron gas is [4]

$$P_0 = -\frac{\partial E_0}{\partial V} = \frac{8\pi m_e^4 c^5}{h^3}\left[\frac{1}{3}x_F^3\sqrt{1 + x_F^2} - f(x_F)\right]. \tag{8}$$

It is almost 1000 times larger than the pressure of the helium nuclei [4]. Further discussions give the relationship between the radius *R* and mass *M* of the star for the relativistically high-density Fermi electron gas

$$\bar{R} = \bar{M}^{2/3}\left[1 - \left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}\right]^{1/2}, \tag{9}$$

where

$$\bar{R} = \left(\frac{2\pi m_e c}{h}\right) R, \tag{10}$$

$$\bar{M} = \frac{9\pi}{8}\frac{M}{m_n}, \tag{11}$$

and

$$\bar{M}_0 = \left(\frac{27\pi}{64\delta}\right)^{3/2}\left(\frac{hc}{2\pi G m_n^2}\right)^{3/2}. \tag{12}$$

In Eq. (18), *G* is the gravitational constant and δ is a parameter of pure number. Some considerations [4] give the upper mass limit M_0 in unit of the mass M_{sun} of our sun

$$M_0 \approx 1.44 M_{sun}, \tag{13}$$

which is also the upper limit for appearance of the white dwarf star.

III. The Correction of The Electron-Electron Interaction For The White Dwarf Star

The ideally Fermi electron gas has been widely discussed in solid state physics. The ground state energy of non-relativistically high-density Fermi electron gas has been calculated by the Hartree-Fock approximation [14,15] and the energy per electron at *T*=0 is

$$\frac{E_{HF}}{N} = \frac{2.21}{r_s^2} - \frac{0.916}{r_s} + 0.0622 \ln r_s - 0.096 \left(\frac{Redberg}{electron}\right), \tag{14}$$

where E_{HF} is the total energy of the Fermi electron gas, and r_s is defined by using the Bohr radius a_B

$$\frac{V}{N} = \frac{4}{3}\pi r_s^3 a_B^3. \tag{15}$$

The first two terms are dominate terms and the ratio of the first term to the second one is proportional r_s or $N^{1/3}$. As N increases, the first term increases faster than the second one. Actually, the calculation of the first term at the right-hand side in Eq. (14) should use Eq. (6) because of the relativistic electrons. Considering x_F>>1 in the relativistic region, then Eq. (6) becomes

$$\frac{E_0}{N} \approx \frac{2\pi m_e^4 c^5}{h^3}\frac{V}{N} x_F^4\left(1 - \frac{4}{3x_F} + \frac{1}{x_F^2}\right). \tag{16}$$

The second term consider the Feynman diagram of the oyster type, so this correlation energy E_1 is [14,15]

$$\begin{aligned}\frac{E_1}{N} &= -\frac{2}{N}\times\frac{1}{2}\times\left[\frac{V}{(2\pi)^3}\right]^2\times\frac{4\pi K_e e^2}{V}\times\frac{16\pi^4}{h^4}\iint_{\vec{p}_1,\vec{p}_2=\vec{0}}^{\vec{p}_F}\frac{d^3\vec{p}_1 d^3\vec{p}_2}{|\vec{p}_1-\vec{p}_2|^2}\\ &= -\frac{3}{2\pi}\left(\frac{2\pi K_e p_F a_B}{h}\right)\left(\frac{e^2}{2a_B}\right) = -\frac{3m_e c K_e e^2}{2h}x_F,\end{aligned} \tag{17}$$

where K_e is the Coulomb's constant. Using Eqs. (16) and (17), the pressure P_{HF} of the Fermi electron gas at $T=0$ is

$$\begin{aligned}P_{HF} &= -\frac{\partial E_{HF}}{\partial V}\\ &= \frac{2\pi m_e^4 c^5}{3h^3}\left(x_F^4 - x_F^2 - 2\frac{2\pi K_e e^2}{hc}x_F^4\right) = \frac{2\pi m_e^4 c^5}{3h^3}\left(x_F^4 - x_F^2 - \frac{2}{137}x_F^4\right),\end{aligned} \tag{18}$$

where $2\pi K_e e^2/hc$ is the fine structure constant [16-19]. It means that the electron-electron interaction causes the pressure about 2/137 time less than the original value.

IV. The Temperature Effect On The Pressure of The Ideal Fermi Electron Gas

The central temperature of a star is usually about 10^7 K, and the upper mass limit in Eq. (13) calculated at T=0 should be improved. Otherwise, it cannot reflect how the relationship between the radius and mass of the white dwarf star varies with temperature. Then we consider the case for T>>0, and the grand partition function in statistical mechanics [4] is

$$q(T,V,z) = \ln Z = \sum_k \ln[1 + z\cdot exp(-\beta E_k)], \tag{19}$$

Where E_k is the kinetic energy, β=1/k_BT, and z=exp($\mu\beta$) with μ the chemical potential of the Fermi electron gas. Since the energy eigenstates are treated as arbitrarily close to each other in a very large volume, the grand partition function becomes

$$\ln Z = \int_0^\infty dEg(E_k)\ln[1 + zexp(-\beta E_k)]. \tag{20}$$

Integrating it is by parts, then we have [4]

$$\ln Z = g\frac{4\pi V}{h^3}\frac{\beta}{3}\int_0^\infty p^3 dp\frac{dE_k}{dp}\frac{1}{z^{-1}exp(\beta E_k) + 1}, \tag{21}$$

where g=2s+1 is the degeneracy factor and

$$p^2 = \frac{E_k^2}{c^2} + 2m_e E_k. \tag{22}$$

Substituting Eq. (22) into Eq. (21), it gives

$$\ln Z = g\frac{4\pi V\beta(2m_e)^{3/2}}{3h^3}\int_0^\infty dE_k\frac{E_k^{3/2}\left[1 + \frac{E_k}{2m_e c^2}\right]^{3/2}}{z^{-1}exp(\beta E_k) + 1}. \tag{23}$$

Using the Taylor series expansion to the first-order term, then we have

$$\ln Z \approx g\frac{4\pi V(2m_e)^{3/2}}{3h^3\beta^{3/2}}\int_0^\infty d(\beta E_k)\frac{(\beta E_k)^{3/2}\left[1 + \frac{3}{2}\left(\frac{1}{2m_e c^2\beta}\right)(\beta E_k)\right]}{z^{-1}exp(\beta E_k) + 1}. \tag{24}$$

It can be written as

$$\ln Z \approx g\frac{4\pi V(2m_e)^{3/2}}{3h^3\beta^{3/2}}\left[\Gamma\left(\frac{5}{2}\right)f_{5/2}(z) + \frac{3}{2}\left(\frac{k_BT}{2m_e c^2}\right)\Gamma\left(\frac{7}{2}\right)f_{7/2}(z)\right], \tag{25}$$

where we define

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty d(\beta E_k)\frac{(\beta E_k)^{n-1}}{z^{-1}e^{(\beta E)} + 1}. \tag{26}$$

The corresponding Fermi energy E_F is roughly 20 MeV [3] and 1/(2$m_e c^2\beta$)~1/1000. The chemical potential μ~E_F so z=$\beta\mu$~20000. When z>>1, the approximation of Eq. (16) [4] is

$$f_n(z) \approx \frac{(\ln z)^n}{n!}, \tag{27}$$

so the ratio of the second term to the first term is

$$\frac{3}{2}\left(\frac{k_BT}{2m_ec^2}\right)\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\frac{f_{7/2}}{f_{5/2}}\approx\frac{3}{2}\left(\frac{1}{1000}\right)\frac{5}{2}\cdot\frac{\ln z}{(7/2)}\approx\frac{1}{100}. \tag{28}$$

According to the relationship ln$Z$=pV/k_BT, the result in Eq. (25) means that the temperature effect explicitly causes the large increase in pressure produced by the high-density Fermi electron gas when T raises from 0 K to 10^7 K. According to Eq. (24), the pressure causing by the Fermi electron gas is

$$P_{\substack{electron\\gas}}\approx\frac{8\pi(2m_e)^{3/2}(k_BT)^{5/2}}{3h^3}\left[\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)+\frac{3}{4}\left(\frac{k_BT}{m_ec^2}\right)\Gamma\left(\frac{7}{2}\right)f_{7/2}(z)\right]. \tag{29}$$

Then we calculate particle number $N(T, V, \mathrm{z})$ using the similar way in statistical mechanics. It gives

$$N(T,V,z)=\mathrm{g}\frac{4\pi V}{h^3}\int_0^\infty p^2dp\frac{1}{z^{-1}exp(\beta E_k)+1}$$

$$=\mathrm{g}\frac{2\pi V(2m_e)^{3/2}}{h^3}\int_0^\infty dE_k\frac{E_k^{1/2}\left(1+\frac{E_k}{2m_ec^2}\right)^{1/2}\left(1+\frac{E_k}{m_ec^2}\right)}{z^{-1}exp(\beta E_k)+1}. \tag{30}$$

Using Taylor series expansion to the first-order term, then we have

$$N(T,V,z)\approx g\frac{2\pi V(2m_e)^{3/2}}{h^3\beta^{3/2}}\int_0^\infty d(\beta E_k)\frac{(\beta E_k)^{1/2}\left[1+\frac{5}{2}\left(\frac{1}{2m_ec^2\beta}\right)(\beta E_k)\right]}{z^{-1}exp(\beta E_k)+1}. \tag{31}$$

Further calculation gives

$$N(T,V,z)\approx\frac{4\pi V(2m_e)^{\frac{3}{2}}(k_BT)^{3/2}}{h^3}\left[\Gamma\left(\frac{3}{2}\right)f_{3/2}(z)+\frac{5}{4}\left(\frac{1}{m_ec^2\beta}\right)\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)\right]. \tag{32}$$

Combing Eq. (29) with Eq. (32), it gives the relationship between $P_{electron\ gas}$, T, V, and N, that is,

$$P_{\substack{electron\\gas}}\approx\frac{2Nk_BT}{3V}\left[\frac{\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)+\frac{3}{4}\left(\frac{k_BT}{m_ec^2}\right)\Gamma\left(\frac{7}{2}\right)f_{7/2}(z)}{\Gamma\left(\frac{3}{2}\right)f_{3/2}(z)+\frac{5}{4}\left(\frac{k_BT}{m_ec^2}\right)\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)}\right]. \tag{33}$$

Further rearrangement gives

$$P_{\substack{electron \\ gas}} \approx \frac{2Nk_BT}{3V}\left[\frac{\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)}{\Gamma\left(\frac{3}{2}\right)f_{3/2}(z)}\right]\left\{\frac{1+\frac{3}{4}\left(\frac{k_BT}{m_ec^2}\right)\left[\frac{\Gamma\left(\frac{7}{2}\right)f_{7/2}(z)}{\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)}\right]}{1+\frac{5}{4}\left(\frac{k_BT}{m_ec^2}\right)\left[\frac{\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)}{\Gamma\left(\frac{3}{2}\right)f_{3/2}(z)}\right]}\right\}$$

$$\approx \frac{Nk_BT}{V}\left(\frac{2}{5}\ln z\right)\left[\frac{1+\frac{3}{4}\left(\frac{k_BT}{m_ec^2}\right)\left(\frac{5}{7}\ln z\right)}{1+\frac{5}{4}\left(\frac{k_BT}{m_ec^2}\right)\left(\frac{3}{5}\ln z\right)}\right]$$

$$\approx \frac{Nk_BT}{V}\left(\frac{2}{5}\ln z\right)\left[1-\frac{3}{14}\left(\frac{k_BT}{m_ec^2}\right)\ln z\right]. \tag{34}$$

It explicitly tells us that the total pressure is mainly proportional to $T/(\ln T)$ at the given particle number N and volume V, and the pressure is higher when temperature increases. The temperature effect causes the pressure is about 4.343 times at 10^8 K higher than 10^7 K.

After obtaining the pressure of the degenerate Fermi electron gas varying with temperature, then we can estimate the relationship between mass and radius of the white dwarf star. The relationship between V and R is

$$V = \frac{4}{3}\pi R^3. \tag{35}$$

Using Eqs (4), (10), (11), and (35), it gives [3]

$$\frac{N}{V} \approx \frac{3M}{8\pi m_n R^3} = \left(\frac{3}{8\pi m_n}\right)\left(\frac{8m_n}{9\pi}\right)\left(\frac{2\pi m_e c}{h}\right)^3\frac{\bar{M}}{\bar{R}^3} = \left(\frac{8\pi m_e^3c^3}{3h^3}\right)\frac{\bar{M}}{\bar{R}^3}. \tag{36}$$

The equilibrium condition [3] is

$$\left(\frac{8\pi m_e^3c^3k_BT}{3h^3}\right)\left(\frac{2}{5}\ln z\right)\left[1-\frac{3}{14}\left(\frac{k_BT}{m_ec^2}\right)\ln z\right]\frac{\bar{M}}{\bar{R}^3} = K'\frac{\bar{M}^2}{\bar{R}^4}, \tag{37}$$

where

$$K' = \frac{\delta}{4\pi}G\left(\frac{8m_n}{9\pi}\right)^2\left(\frac{2\pi m_e c}{h}\right)^4. \tag{38}$$

In Eq. (38), δ is a parameter of pure number and G is the gravitational constant [4]. Substituting Eq. (38) into Eq. (37), then we have

$$\frac{\bar{M}}{\bar{R}} = \frac{4\pi k_B T}{\delta G}\left(\frac{27}{64m_n^2}\right)\left(\frac{h}{2\pi m_e c}\right)\left(\frac{2}{5}\ln z\right)\left[1-\frac{3}{14}\left(\frac{k_B T}{m_e c^2}\right)\ln z\right]$$

$$= \left(\frac{27}{64\delta G}\right)\left(\frac{2h}{m_n^2 m_e c}\right)\left(\frac{2}{5}k_B T\ln z\right)\left[1-\frac{3}{14}\left(\frac{k_B T}{m_e c^2}\right)\ln z\right]. \qquad (39)$$

Further rearrangement gives

$$\frac{M}{R} = \left(\frac{3}{2\delta G m_n}\right)\left(\frac{2}{5}k_B T\ln z\right)\left[1-\frac{3}{14}\left(\frac{k_B T}{m_e c^2}\right)\ln z\right]. \qquad (40)$$

It explicitly tells us that the relationship between M and R depends on T and m_n. For example, a white dwarf star with mass $M=M_{sun}$=1.99x10^{30} kg and a radius R=6.371x10^6 m the same as earth is used in Eq. (40). Then some constants [20], G=6.67259x10^{-11} $m^{-3}kg^{-1}s^{-2}$, 1.67493x10^{-27} kg, k_B=1.38066x10^{-23} JK^{-1}, T=1.16x10^7 K, and z=20000, are also substituted into Eq. (40), then we have

$$\frac{M}{R} = \frac{8.48\times10^{21}}{\delta} = 3.12\times10^{23}, \qquad (41)$$

which gives δ=0.02716. This is a reasonable value and Eq. (40) can help us to estimate the inner temperature of a white dwarf star.

V. Conclusion

In summary, the calculation from statistical mechanics shows that the temperature effect is at 10^7 K and the ideally degenerate Fermi electron gas has to be corrected at high temperature. First the electron-electron interaction is considered at T=0. The calculation considers the relativistic electrons and the result shows that this effect causes the pressure is 2/137 time less than the original value. Because the electron gas is at very high temperature about 10^7 K, the temperature effect has to be considered and the pressure needs to be calculated by statistical mechanics. Then from the deduction, the pressure produced by the Fermi electron gas is mainly proportional to $T/(\ln T)$ at the given particle number N and volume V, and the pressure is 4.343 times at 10^8 K higher than it at 10^7 K. Finally, from the equilibrium condition, we obtain the relationship between M and R which depends on T and m_n. An example of a dwarf star uses the mass of our sun and the radius of Earth to calculate an unknown parameter δ=0.02716. This relationship can help us to estimate the inner temperature of a white dwarf star.

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