

SOME FINITE SERIES AND THEIR APPLICATION

(Saikat sarkar) (Undergraduate student of visva-bharati University)

Q.1. Find the sum of $\binom{2n}{0}^2 - \binom{2n}{1}^2 + \binom{2n}{2}^2 - \dots - \binom{2n}{2n-1}^2 + \binom{2n}{2n}^2$.

Answer: we know that $(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \dots + \binom{2n}{2n}x^{2n}$ (1)

and $(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \dots + \binom{2n}{2n}x^{2n}$ (2)

we can write $\binom{2n}{0}^2 + \binom{2n}{1}^2 + \binom{2n}{2}^2 + \dots + \binom{2n}{2n}^2 = \binom{2n}{2n} - \binom{2n}{2n-1}\binom{2n}{1} + \binom{2n}{2n-2}\binom{2n}{2} - \dots - \binom{2n}{1}\binom{2n}{2n-1} + \binom{2n}{2n}$. so, the coefficient of x^{2n} in $(1+x)^{2n} \cdot (1-x)^{2n}$.

$$= \text{the coefficient of } x^{2n} \text{ in } (1-x^2)^{2n}.$$

using binomial expansion we get the coefficient of x^{2n} in $(1-x^2)^{2n}$ and that is $(-1)^n \binom{2n}{n}$.

Q.2. Let $S_n, n \geq 1$, be the sets defined as follows : $S_1 = \{0\}, S_2 = \left\{\frac{3}{2}, \frac{5}{2}\right\}, S_3 = \left\{\frac{8}{3}, \frac{11}{3}, \frac{14}{3}\right\}, S_4 = \left\{\frac{15}{4}, \frac{19}{4}, \frac{23}{4}, \frac{27}{4}\right\}$ and so on. Then, find the sum of the elements of S_{20} .

Answer: From the pattern of the sets we get, $S_{20} = \left\{\frac{399}{20}, \frac{419}{20}, \dots, \frac{779}{20}\right\}$. Because the first term

of every set is $\frac{n^2-1}{n}, n \geq 1$. so, the sum of the all elements in S_{20} is $\frac{20 \left\{\frac{399+779}{20}\right\}}{2}$.

$$= \frac{1178}{2}.$$

$$= 589.$$

Q.3. For any positive integer k , compute the integral part of $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{10,000}}$

using the property that $2(\sqrt{k+1} - \sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1})$.

Answer: we know that $2(\sqrt{k+1} - \sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1}), k = 1, 2, \dots$

after putting $k = 2$ we get, $2(\sqrt{3} - \sqrt{2}) < \frac{1}{\sqrt{2}} < 2(\sqrt{2} - 1)$.

after putting $k = 3$ we get, $2(\sqrt{4} - \sqrt{3}) < \frac{1}{\sqrt{3}} < 2(\sqrt{3} - \sqrt{2})$.

⋮ ⋮ ⋮

after putting $k = 10,000$ we get, $2(\sqrt{10,001} - \sqrt{10,000}) < \frac{1}{\sqrt{10,000}} < 2(\sqrt{10,000} - \sqrt{9999})$.

after adding all inequalities we get, $2(\sqrt{10,001} - \sqrt{2}) < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{10,000}} < 2(\sqrt{10,000} - 1)$.

$$\text{or, } 197.181 < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{10,000}} < 198.$$

so, the integral part is 197.

Q.4. Find the sum of the series of $1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta$.

Answer: we know that $(1 + a)^n$

$$= 1 + \binom{n}{1} a + \binom{n}{2} a^2 + \dots + \binom{n}{n} a^n, \text{ where } a \text{ is real number or complex number.}$$

if we put $a = \cos \theta + i \sin \theta$ we get, $1 + \binom{n}{1}(\cos \theta + i \sin \theta) + \binom{n}{2}(\cos \theta + i \sin \theta)^2 + \dots +$

$$\binom{n}{n}(\cos \theta + i \sin \theta)^n.$$

so, this implies that $\{(1 + \cos \theta) + i \sin \theta\}^n = 1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta + i \binom{n}{1} \sin \theta + i \binom{n}{2} \sin 2\theta + \dots + i \binom{n}{n} \sin n\theta \dots (1)$

and if we put $a = \cos \theta - i \sin \theta$ we get, $\{(1 + \cos \theta) - i \sin \theta\}^n$

$$= 1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta - i \binom{n}{1} \sin \theta - i \binom{n}{2} \sin 2\theta - \dots - i \binom{n}{n} \sin n\theta. \dots (2)$$

after adding (1) and (2) we get, $\{(1 + \cos \theta) + i \sin \theta\}^n + \{(1 + \cos \theta) - i \sin \theta\}^n = 2\{1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta\}$

or, $\left\{2 \cos^2 \frac{\theta}{2} + i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right\}^n + \left\{2 \cos^2 \frac{\theta}{2} - i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right\}^n = 2\{1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta\}$.

or, $\left(\cos \frac{\theta}{2}\right)^n \left\{2 \cos \frac{\theta}{2} + i 2 \sin \frac{\theta}{2}\right\}^n + \left(\cos \frac{\theta}{2}\right)^n \left\{2 \cos \frac{\theta}{2} - i 2 \sin \frac{\theta}{2}\right\}^n = 2\{1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta\}$.

or, $(2 \cos \frac{\theta}{2})^n \cos \frac{n\theta}{2} = \{1 + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta\}$.

Q.5. Find the value of $\sum_{r=1}^{10} (r^2 + 1)r!$.

Answer: we know that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + \dots + n \cdot n! = (n + 1)! - 1 \dots (1)$

so, $\sum_{r=1}^{10} (r^2 + 1)r! = \sum_{r=1}^{10} (r + 1)^2 \cdot r! - 2 \sum_{r=1}^{10} r \cdot r!$

$$= \sum_{r=1}^{10} (r + 1) \cdot (r + 1)! - 2 \sum_{r=1}^{10} r \cdot r! = 12! - 1 - 2(11! - 1) -$$

1 [using the property (1)].

$$= 12! - 2 \cdot 11! = 10 \cdot 11!$$

Q.6. If $S_n = \sum_{r=0}^n \frac{1}{\binom{n}{r}}$ and $t_n = \sum_{r=0}^n \frac{r}{\binom{n}{r}}$, then find the value of $\frac{t_n}{S_n}$.

Answer: so, $t_n = \sum_{r=0}^n \frac{r}{\binom{n}{r}}$.

$$\begin{aligned} \text{or, } t_n &= -\sum_{r=0}^n \frac{-r}{\binom{n}{r}} \\ &= -\sum_{r=0}^n \frac{n-r-n}{\binom{n}{r}} \\ &= -\sum_{r=0}^n \frac{n-r}{\binom{n}{n-r}} + \sum_{r=0}^n \frac{n}{\binom{n}{r}} \\ &= -t_n + nS_n. \end{aligned}$$

$$\text{or, } t_n = -t_n + nS_n.$$

$$\text{or, } \frac{t_n}{S_n} = \frac{n}{2}.$$

Q.7. Compute the sum : $S_n = \sum_{k=0}^n \frac{1}{(k+1)(k+2)} \binom{n}{k}$.

Answer: we know that $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$.

after integrating both side with respect to x we get, $\int (1+x)^n dx = \int \{1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n\}$

$$\text{or, } \frac{(1+x)^{n+1}}{n+1} = x + \binom{n}{1} \frac{x^2}{2} + \binom{n}{3} \frac{x^3}{3} + \dots + \binom{n}{n+1} \frac{x^{n+1}}{n+1} + c_1, \text{ where } c_1 \text{ is integrating constant.}$$

To find the value of c_1 we have to put $x = 0$ and we get, $c_1 = \frac{1}{n+1}$.

$$\text{Now, } \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = x + \binom{n}{1} \frac{x^2}{2} + \binom{n}{3} \frac{x^3}{3} + \dots + \binom{n}{n+1} \frac{x^{n+1}}{n+1} \dots (1)$$

$$\begin{aligned} \text{Now, after integrating both side again we get, } & \frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{x^2}{1 \cdot 2} \binom{n}{0} + \binom{n}{1} \frac{x^3}{2 \cdot 3} + \binom{n}{3} \frac{x^4}{3 \cdot 4} + \\ & \dots + \binom{n}{n} \frac{x^{n+2}}{(n+1) \cdot (n+2)} + c_2 \dots (2) \end{aligned}$$

$$\text{putting } x = 0 \text{ we get, } c_2 = \frac{1}{(n+1)(n+2)}.$$

$$\begin{aligned} \text{putting the value of } c_2 \text{ in the equation (2), } & \sum_{k=0}^n x^{k+2} \binom{n}{k} \cdot \frac{1}{(k+1)(k+2)} = \frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1} - \\ & \frac{1}{(n+1)(n+2)} \dots (3) \end{aligned}$$

$$\text{putting } x = 1 \text{ in equation (3) we get, } \sum_{k=0}^n \binom{n}{k} \cdot \frac{1}{(k+1)(k+2)} = \frac{2^{n+2}}{(n+1)(n+2)} - \frac{1}{n+1} - \frac{1}{(n+1)(n+2)}.$$

$$= \frac{2^{n+2} - (n+3)}{(n+1)(n+2)}.$$

: Reference:

- *Test of Mathematics at the 10 + 2 Level – INDIAN STATISTICAL INSTITUTE*