News Limit Formulas for Exponential of the Digamma Function, k-Power and Exponential Function, Involving Gamma Function and Pochhammer Symbol

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Abstract. We derive some identities for limit of exponential for digamma function, k-power and exponential function, involving gamma functions and Pochhammer symbols.

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1. Introduction

In present paper, we derive some identities for limit of exponential for digamma function, k-power and exponential function, involving gamma functions and Pochhammer symbols, such as,

$$\exp[\psi(x+1)+\gamma] = \lim_{m\to\infty}\left\{\frac{\Gamma\left(\frac{1}{2}\left(2+x-\sqrt{x^2+\frac{4x}{m}}\right)\right)\Gamma\left(\frac{1}{2}\left(2+x+\sqrt{x^2+\frac{4x}{m}}\right)\right)}{\Gamma(x+1)}\right\}^m,$$

$$x^k = \lim_{n\to\infty}\frac{(xn)_k}{(n)_k},$$

and

$$e^x = \lim_{n\to\infty}\frac{(n^2)_{(x+1)n}}{(n^2)_{xn}(n^2)_n}.$$

2. The Limit Formula for Exponential of Digamma Function

Theorem 2.1. *We have*

$$\exp[\psi(x+1)+\gamma] = \lim_{m\to\infty}\left\{\frac{\Gamma\left(\frac{1}{2}\left(2+x-\sqrt{x^2+\frac{4x}{m}}\right)\right)\Gamma\left(\frac{1}{2}\left(2+x+\sqrt{x^2+\frac{4x}{m}}\right)\right)}{\Gamma(x+1)}\right\}^m, \tag{2.1}$$

where $\exp(x)$ *denotes the expoential function,* $\Gamma(x)$ *denotes the gamma function,* $\psi(x)$ *denotes the digamma function and* γ *denotes the Euler-Mascheroni constant.*

Proof. We know [1] the infinite sum representation for digamma function, given by

$$\begin{aligned} \psi(x+1)+\gamma &= \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right) \\ &= \sum_{k=1}^{\infty}\log\left[\exp\left(\frac{1}{k}-\frac{1}{x+k}\right)\right] \\ \Rightarrow \exp[\psi(x+1)+\gamma] &= \prod_{k=1}^{\infty}\exp\left[\frac{x}{k(x+k)}\right], \end{aligned} \tag{2.2}$$

for $\operatorname{Re}(x)>0$.

On the other hand, we know [2, p. 5, Corollary 7] the limit formula for exponential function given by

$$\exp(z)=\lim_{m\to\infty}\left(1-\frac{z}{m}\right)^{-m}. \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$\begin{aligned} \exp[\psi(x+1)+\gamma] &= \lim_{m\to\infty}\prod_{k=1}^{\infty}\left[1-\frac{x}{km(x+k)}\right]^{-m} \\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[1-\frac{x}{km(x+k)}\right]^{-1}\right\}^{m} \\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[\frac{km(x+k)-x}{km(x+k)}\right]^{-1}\right\}^{m} \\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[\frac{km(x+k)}{km(x+k)-x}\right]\right\}^{m} \\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[\frac{1}{1-\frac{x}{km(x+k)}}\right]\right\}^{m} \\ &= \lim_{m\to\infty}\left\{\frac{\Gamma\left(\frac{1}{2}\left(2+x-\sqrt{x^2+\frac{4x}{m}}\right)\right)\Gamma\left(\frac{1}{2}\left(2+x+\sqrt{x^2+\frac{4x}{m}}\right)\right)}{\Gamma(x+1)}\right\}^{m}, \end{aligned}$$

which is the desired result. □

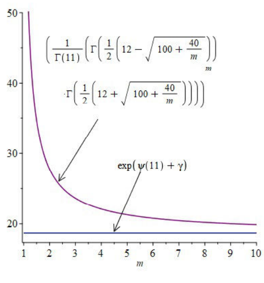

Figure 2.1. For $x=10$ in (2.1).

3. The k–th Power

Theorem 3.1. *We have*

$$x^k = \lim_{n\to\infty} \frac{\Gamma(k+xn)\Gamma(n)}{\Gamma(k+n)\Gamma(xn)}, \tag{3.1}$$

where $\Gamma(x)$ *denotes the gamma function.*

Proof. In [3], we find the Stirling's approximation formula given below

$$\Gamma(n+1) = \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Gamma(k+xn)\Gamma(n)}{\Gamma(k+n)\Gamma(xn)} &= \lim_{n\to\infty} \frac{\sqrt{2\pi(k+xn)}\left(\frac{k+xn}{e}\right)^{k+xn}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(k+n)}\left(\frac{k+n}{e}\right)^{k+n}\sqrt{2\pi xn}\left(\frac{xn}{e}\right)^{xn}} \\
&= \lim_{n\to\infty}\left(\frac{k+xn}{xk+xn}\right)^{1/2} \cdot \frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}} \cdot \frac{\left(\frac{1}{e}\right)^{k+xn}\left(\frac{1}{e}\right)^n}{\left(\frac{1}{e}\right)^{k+n}\left(\frac{1}{e}\right)^{xn}} \\
&= \lim_{n\to\infty}\left[\frac{k+xn}{x\,(k+n)}\right]^{1/2} \cdot \frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}} \\
&= \frac{1}{x^{1/2}} \lim_{n\to\infty}\left(\frac{k+xn}{k+n}\right)^{1/2} \cdot \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] \\
&= \frac{1}{x^{1/2}} \lim_{n\to\infty}\left[\frac{xn\left(1+\frac{k}{xn}\right)}{n\left(1+\frac{k}{n}\right)}\right]^{1/2} \cdot \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] \\
&= \frac{1}{x^{1/2}} \lim_{n\to\infty}\left[\frac{x\left(1+\frac{k}{xn}\right)}{1+\frac{k}{n}}\right]^{1/2} \cdot \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] \\
&= \frac{1}{x^{1/2}} \cdot x^{1/2} \cdot \lim_{n\to\infty}\left(\frac{1+\frac{k}{xn}}{1+\frac{k}{n}}\right)^{1/2} \cdot \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] \\
&= 1 \cdot \left[\lim_{n\to\infty}\left(\frac{1+\frac{k}{xn}}{1+\frac{k}{n}}\right)\right]^{1/2} \cdot \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] \\
&= 1 \cdot 1^{1/2} \cdot \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] \\
&= \lim_{n\to\infty}\left[\frac{(k+xn)^{k+xn}n^n}{(k+n)^{k+n}(xn)^{xn}}\right] = \lim_{n\to\infty}\left[\frac{\left(1+\frac{k}{xn}\right)^{k+xn}n^n(xn)^{k+xn}}{\left(1+\frac{k}{n}\right)^{k+n}(xn)^{xn}n^{k+n}}\right] \\
&= \lim_{n\to\infty}\left[\frac{\left(1+\frac{k}{xn}\right)^{k+xn}}{\left(1+\frac{k}{n}\right)^{k+n}}\right] \cdot \lim_{n\to\infty}\left[\frac{n^n(xn)^{k+xn}}{(xn)^{xn}n^{k+n}}\right] \\
&= \lim_{n\to\infty}\left[\frac{\left(1+\frac{k}{xn}\right)^k\left(1+\frac{k}{xn}\right)^{xn}}{\left(1+\frac{k}{n}\right)^k\left(1+\frac{k}{n}\right)^n}\right] \cdot \lim_{n\to\infty}\left[\frac{n^n(xn)^k(xn)^{xn}}{(xn)^{xn}n^kn^n}\right] \\
&= \left[\frac{\lim\limits_{n\to\infty}\left(1+\frac{k}{xn}\right)^k \lim\limits_{n\to\infty}\left(1+\frac{k}{xn}\right)^{xn}}{\lim\limits_{n\to\infty}\left(1+\frac{k}{n}\right)^k \lim\limits_{n\to\infty}\left(1+\frac{k}{n}\right)^n}\right] \cdot \lim_{n\to\infty}\left(\frac{n^nx^kn^k}{n^kn^n}\right) = \frac{1^k \cdot e^k}{1^k \cdot e^k} \cdot x^k = x^k,
\end{aligned}$$

which is the desired result. □

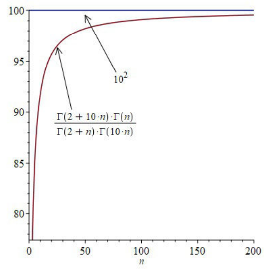

Figure 3.1. For $x=10$ and $k=2$ in (3.1).

Corollary 3.2. *We have*

$$x^k = \lim_{n\to\infty} \frac{(xn)_k}{(n)_k}, \tag{3.3}$$

where $(x)_k$ *denotes the Pochhammer symbol.*

Proof. First, let the following expansion

$$x^k = \lim_{n\to\infty} \frac{\Gamma(k+xn)\Gamma(n)}{\Gamma(k+n)\Gamma(xn)} = \lim_{n\to\infty}\left[\frac{\Gamma(k+xn)}{\Gamma(xn)}\cdot\frac{\Gamma(n)}{\Gamma(k+n)}\right] = \lim_{n\to\infty}\left[\frac{\frac{\Gamma(k+xn)}{\Gamma(xn)}}{\frac{\Gamma(k+n)}{\Gamma(n)}}\right]. \tag{3.4}$$

The Pochhammer symbol may be defined by [4]

$$(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)} \tag{3.5}$$

for $n \geqq 0$.

From (3.4) and (3.5), it follows that

$$x^k = \lim_{n\to\infty} \frac{(xn)_k}{(n)_k},$$

which is the desired result. □

4. LIMIT FORMULA FOR EXPONENTIAL FUNCTION

Theorem 4.1. *We have*

$$e^x = \lim_{n\to\infty} \frac{((n+x)\,n)_n}{(n^2)_n}, \tag{4.1}$$

where e^x *denotes the exponential function and* $(x)_n$ *denotes the Pochhammer symbol.*

Proof. The exponential function [5, p. 156] can be defined as the following limit:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\frac{(n+x)^n}{n^n}. \tag{4.2}$$

From Corollary 3.2 and (4.2), it follows that

$$e^x = \lim_{n\to\infty}\lim_{N\to\infty}\frac{\frac{((n+x)\,N)_n}{(N)_n}}{\frac{(nN)_n}{(N)_n}} = \lim_{n\to\infty}\lim_{N\to\infty}\frac{((n+x)\,N)_n}{(nN)_n} = \lim_{n\to\infty}\frac{((n+x)\,n)_n}{(n^2)_n},$$

which is the desired result. □

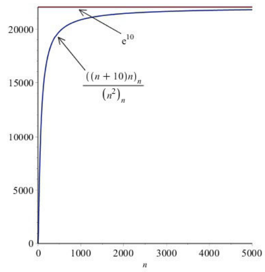

Figure 4.1. For $x = 10$ in (5.1).

Corollary 4.2. *We have*

$$e^x = \lim_{n\to\infty} \frac{(n^2)_{(x+1)n}}{(n^2)_{xn}(n^2)_n}, \tag{4.3}$$

where e^x *denotes the exponential function and* $(x)_n$ *denotes the Pochhammer symbol.*

Proof. In [6, p. 239, (I.4)], we find the formula in general

$$(a + kn)_n = \frac{(a)_{(k+1)n}}{(a)_{kn}}. \tag{4.4}$$

Let $a \to n^2$ and $k \to x$ into (4.4) and encounter

$$((n + x)\, n)_n = \frac{(n^2)_{(x+1)n}}{(n^2)_{xn}}. \tag{4.5}$$

From Theorem 4.1 and (4.5), it follows that

$$e^x = \lim_{n\to\infty} \frac{(n^2)_{(x+1)n}}{(n^2)_{xn}(n^2)_n},$$

which is the desired result. □

5. Some Exercises

Exercise 5.1. Prove that

$$e^{xy} = \lim_{n\to\infty} \frac{(n^2)_{(x+y)n}}{(n^2)_{xn}(n^2)_{yn}}.$$

Exercise 5.2. Prove that

$$\psi(nx) - \psi(n) = \lim_{\epsilon\to 0}\left[\frac{(xn)_\epsilon}{(n)_\epsilon \cdot \epsilon} - \frac{1}{\epsilon}\right].$$

References

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