A simple derivation of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac statistics for systems in a heat bath.

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Abstract

We derive these laws with a simplicity for high school students and cocky beach girls.

1 Maxwell-Boltzmann.

Consider a system of N particles in contact with a heat reservoir; the system can be in a finite number of states $|i\rangle$ each characterized by an energy ϵ_i and other numbers α_i. The question is, given a temperature T, what is the probability that it is found in $|i\rangle$? It is very simple to figure out the answer, given that we have to allow for the system to change from $|i\rangle$ to $|j\rangle$ with a probability $p(i \to j)(t)$ per unit time. These probabilities have to be determined dynamically; suppose initial probability $p_i(t)$ is given. The latter satisfies

$$\dot{p}_i(t) = \sum_j (p_j(t)p(j \to i)(t) - p_i(t)p(i \to j)(t))$$

where t is time. One notices that

$$\sum_i \dot{p}_i(t) = \sum_{i,j} (p_j(t)p(j \to i)(t) - p_i(t)p(i \to j)(t)) = 0$$

so that total probability is conserved. Moreover, it is clear that

$$p(i \to j)p(j \to k) = \kappa p(i \to k)$$

must be j independent and κ (i,k) independent as a matter of "homology" condition. It signifies that, in a way, the transition from i to k can happen in multiple stages where only the number of intermediate stages matters and not the detais thereof. The stronger form sets κ equal to one, which would mean that a system has its own radiative temperature and

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cannot be heated up, by means of a reservoir, to a stable state of higher temperature. This law can be derived in another way by noticing that

$$p(i \to j) = p_{i^\star} p_j$$

where p_i has some functional form in terms of ϵ_i and α_i and $p_{i^\star}$ is the fictional probability of destroying a state $|i\rangle$. This means that the probability of transition is given by the probability of destruction of a state $|i\rangle$ followed by the birth of $|j\rangle$ where both these happenings are independent of one and another. Obviously, this is achieved by putting ϵ_i, α_i to $-\epsilon_i, -\alpha_i$ and one remarks that

$$\dot{p}_i(t) = (\sum_j p_j(t) p_{j^\star}(t) - p_{i^\star}(t)) p_i(t).$$

Now, an energy ϵ_i reflects a certain wavelength λ_i with $\epsilon_i = \frac{\hbar c}{\lambda_i}$ and c the speed of light, $\hbar$ the quantum constant. $\tau_i = \frac{\lambda_i}{c} = \frac{\hbar}{\epsilon_i}$ is a timescale associated to observation of that energy. A suitable defining characteristic is therefore

$$0 = p_i(\frac{\hbar}{\epsilon_i}) - p_i(0) = \int_0^{\frac{\hbar}{\epsilon_i}} \dot{p}_i(t) dt = \int_0^{\frac{\hbar}{\epsilon_i}} (\sum_j p_j(t) p_{j^\star}(t) - p_{i^\star}(t)) p_i(t) dt.$$

For example, $p_i(0) = 1$ and $p_j(0) = 0$ with $i \neq j$ satisfies this criterium although $p_i(\tau) = 1$ for all $\tau > 0$ and therefore this timescale is rather ambiguous. It is utterly clear, given that the system can only “sing” the modes ϵ_i that this requirement signifies that it is stable on the associated timescale. This definition of temporary temperature $T_{[0,\tau_i]}$ associated to a time interval of measurement, for example by putting your finger on a heating plate, is now open for discussion and weakening. Now, we come back to the homology law which would imply that

$$p_{j^\star} p_j = \kappa$$

for all j and therefore

$$\dot{p}_i(t) = N \kappa p_i(t) - \kappa.$$

This leads for a constant κ to solutions of the kind

$$p_i(t) = \frac{1}{N}(1 - e^{N\kappa t})$$

which only retains $\kappa = -\infty$ which is the uniform distribution wich is in conflict with $p_{j^\star} p_j = \kappa$. This suggests one to replace

$$p(i \to j) p(j \to k) = \kappa p(i \to k)$$

with

$$\sum_j p(i \to j) p(j \to k) = \kappa p(i \to k)$$

which is a principle of ignorance indicating that one does not know the intermediate state as a matter of principle. This leads to

$$\sum_j p_{j^\star} p_j = \kappa$$

and therefore

$$\dot{p}_i(t) = (\kappa - p_{i^\star}(t)) p_i(t).$$

This at least incorporates the case $p_i = 1, p_j = 0$ given the functionality $p_k = \delta(\epsilon_k - \epsilon_i)$.

Traditional Maxwell-Boltzmann can be derived from one principle; in general, one looks for a time independent distribution with the property that it factorizes over different independent subsystems. That is

$$p_{1\cup 2}(\epsilon_i^1 + \epsilon_j^2) = \sum_{(k,l):\epsilon_k^1+\epsilon_l^2=\epsilon_i^1+\epsilon_j^2} p_1(\epsilon_k^1) p_2(\epsilon_l^1)$$

and $p(\epsilon_i)$ where ϵ_i is a time independent energy. This implies

$$p(\epsilon_i) = \frac{e^{-\beta\epsilon_i}}{\sum_j e^{-\beta\epsilon_j}}.$$

Here, $\beta = \frac{1}{k_B T}$ where k_B is Boltzmann constant and T the temperature in kelvin. This is *not* a solution to our above system which requires different $p(i \to j)$ violating the homology condition. The separability condition is often assumed to be correct although it excludes hidden correlations between both subsystems by means of interaction through the heat bath. Such equilibrium will never settle as is most easily seen .