

Which Rows in Pascal's Triangle Sum to Perfect Numbers?

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Abstract: In this note we show which rows in Pascal's Triangle sum to Perfect Numbers. We end this note with a conjectured algorithm allowing us to calculate all the proper divisors for any Perfect Number greater than 6.

Starting with Pascal's triangle we find the horizontal numbers sum to powers of 2

$$\begin{array}{cccccc}
 & & & & & 1 & & & & & 2^0 \\
 & & & & & & 1 & & & 1 & & 2^1 \\
 & & & & & 1 & & 2 & & 1 & & 2^2 \\
 & & & & 1 & & 3 & & 3 & & 1 & & 2^3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & 2^4
 \end{array}$$

Next, a perfect number is a number that is equal to the sum of its proper divisors, but excluding the number itself. For example, 6 is a perfect number since $6 = 1 + 2 + 3$. With p some prime Euclid proved that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is a Mersenne prime, and Euler proved that all even perfect numbers have this form for some p . It is unknown if there are infinitely many even perfect numbers or any *odd* perfect numbers.

To see which rows in Pascal's triangle sum to perfect numbers we have the following:

$$2^1(2^2 - 1) = 6 = 2^1 + 2^2$$

$$2^2(2^3 - 1) = 28 = 2^2 + 2^3 + 2^4$$

$$2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$

$$2^6(2^7 - 1) = 8128 = 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}$$

Theorem:

$$2^{p-1}(2^p - 1) = 2^{p-1} + 2^p + 2^{p+1} + \dots + 2^{2(p-1)}$$

Proof:

We use the fact that

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^m = 2^{m+1} - 1$$

Therefore,

$$\begin{aligned} & (1 + 2^1 + 2^2 + 2^3 + \dots + 2^{2(p-1)}) - (1 + 2^1 + 2^2 + 2^3 + \dots + 2^{p-2}) \\ &= 2^{2(p-1)+1} - 1 - (2^{(p-2)+1} - 1) = 2^{2p-1} - 2^{p-1} = 2^p(2^{p-1}) - 2^{p-1} \\ &= 2^{p-1}(2^p - 1) \end{aligned}$$

□

The above allows us to easily calculate the proper divisors for any perfect number, which we will be left as a conjecture.

Conjecture: For $p > 2$ the proper divisors for any perfect number $2^{p-1}(2^p - 1)$ are given by

$$\begin{aligned} 2^{p-1}(2^p - 1) &= 2^{p-1} + (1 + (2^p - 1)) + (2 + (2^{p+1} - 2)) + (2^2 + (2^{p+2} - 2^2)) \\ &\quad + (2^3 + (2^{p+3} - 2^3)) + \dots + (2^{p-2} + (2^{2(p-1)} - 2^{p-2})) \end{aligned}$$

For example,

$$\begin{aligned} 2^4(2^5 - 1) &= 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8 \\ &= 16 + (1 + 31) + (2 + 62) + (4 + 124) + (8 + 248) \\ &= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 \end{aligned}$$