Which Rows in Pascal's Triangle Sum to Perfect Numbers?

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Abstract: In this note we show which rows in Pascal's Triangle sum to Perfect Numbers. We end this note with a conjectured algorithm allowing us to calculate all the proper divisors for any Perfect Number greater than 6.

Starting with Pascal's triangle we find the horizontal numbers sum to powers of 2

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & 2^0 \\
 & & & 1 & & 1 & & & & & 2^1 \\
 & & 1 & & 2 & & 1 & & & & 2^2 \\
 & 1 & & 3 & & 3 & & 1 & & & 2^3 \\
1 & & 4 & & 6 & & 4 & & 1 & & 2^4
\end{array}
$$

Next, a perfect number is a number that is equal to the sum of its proper divisors, but excluding the number itself. For example, 6 is a perfect number since $6 = 1 + 2 + 3$. With p some prime Euclid proved that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is a Mersenne prime, and Euler proved that all even perfect numbers have this form for some p. It is unknown if there are infinitely many even perfect numbers or any *odd* perfect numbers.

To see which rows in Pascal's triangle sum to perfect numbers we have the following:

$$2^1(2^2 - 1) = 6 = 2^1 + 2^2$$

$$2^2(2^3 - 1) = 28 = 2^2 + 2^3 + 2^4$$

$$2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$

$$2^6(2^7 - 1) = 8128 = 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}$$

Theorem:

$$2^{p-1}(2^p - 1) = 2^{p-1} + 2^p + 2^{p+1} + \cdots + 2^{2(p-1)}$$

Proof:

We use the fact that

$$1 + 2^1 + 2^2 + 2^3 + \ \dots + 2^m = 2^{m+1} - 1$$

Therefore,

$$\begin{aligned}&\left(1 + 2^1 + 2^2 + 2^3 + \ \dots + 2^{2(p-1)}\right) - (1 + 2^1 + 2^2 + 2^3 + \ \dots + 2^{p-2})\\ &= 2^{2(p-1)+1} - 1 - \left(2^{(p-2)+1} - 1\right) = 2^{2p-1} - 2^{p-1} = 2^p(2^{p-1}) - 2^{p-1}\\ &= 2^{p-1}(2^p - 1)\end{aligned}$$

□

Is there a systematic and unique way to add the numbers from the relevant rows for a particular perfect number in Pascal's triangle showing the proper divisors of that perfect number? For example, from rows 1 and 2 in Pascal's triangle we have $6 = 1 + 2 + (1 + 1 + 1)$. For all other perfect numbers we conjecture the following:

Conjecture: For $p > 2$ the proper divisors for any perfect number $2^{p-1}(2^p - 1)$ are given by

$$\begin{aligned}2^{p-1}(2^p - 1) = 2^{p-1} &+ \left(1 + (2^p - 1)\right) + \left(2 + (2^{p+1} - 2)\right) + \left(2^2 + (2^{p+2} - 2^2)\right)\\ &+ \left(2^3 + (2^{p+3} - 2^3)\right) + \cdots + \left(2^{p-2} + \left(2^{2(p-1)} - 2^{p-2}\right)\right)\end{aligned}$$

For example,

$$\begin{aligned}&2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8\\ &= 16 + (1 + 31) + (2 + 62) + (4 + 124) + (8 + 248)\\ &= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248\end{aligned}$$