Pascal's Triangle and Perfect Numbers as Sums of Powers of 2

By Robert Spoljaric

Abstract: In this note we show that the even Perfect Numbers can be found in Pascal's Triangle by expressing the even Perfect Numbers as sums of powers of 2

The horizontal numbers in Pascal's triangle sum to powers of 2

1	2^0
1 1	2^1
1 2 1	2^2
1 3 3 1	2^3
1 4 6 4 1	2^4

A perfect number is a number that is equal to the sum of its proper divisors, but excluding the number itself. For example, 6 is a perfect number since $6 = 1 + 2 + 3$. With p some prime Euclid proved that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is a Mersenne prime, and Euler proved that all even perfect numbers have this form for some p. It is unknown if there are infinitely many even perfect numbers or any *odd* perfect numbers.

To show how the even perfect numbers can be found in Pascal's triangle we have only to see the following

$$2^1(2^2 - 1) = 6 = 2^1 + 2^2$$

$$2^2(2^3 - 1) = 28 = 2^2 + 2^3 + 2^4$$

$$2^4(2^5 - 1) = 496 = 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$

$$2^6(2^7 - 1) = 8128 = 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}$$

Theorem:

$$2^{p-1}(2^p - 1) = 2^{p-1} + 2^p + 2^{p+1} + \cdots + 2^{2(p-1)}$$

Proof:

We use the fact that

$$1 + 2^1 + 2^2 + 2^3 + \ \dots + 2^m = 2^{m+1} - 1$$

Therefore,

$$\left(1 + 2^1 + 2^2 + 2^3 + \ \dots + 2^{2(p-1)}\right) - \left(1 + 2^1 + 2^2 + 2^3 + \ \dots + 2^{p-2}\right)$$
$$= 2^{2(p-1)+1} - 1 - \left(2^{(p-2)+1} - 1\right) = 2^{2p-1} - 2^{p-1} = 2^p(2^{p-1}) - 2^{p-1}$$
$$= 2^{p-1}(2^p - 1)$$

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